CHAPTER 5

Tensors and the Equations of Fluid Motion

We have seen that there are a whole range of things that we can represent on the computer. We have solved some simple problems such as Laplace's equation on a unit square at the origin in the first quadrant. From the description of the problem, you can see that it was really a very specific problem. Our objective was to dive into the process of representing and solving partial differential equations on the computer and we have achieved that objective. The problem was simple enough that we could analyse the resulting discrete equations. Similarly, we have studied the first order one-dimensional linear wave equation. We have also seen the heat equation and the quasi linear version of the wave equation. We have some basic understanding of the issues involved in representing and solving these problems on the computer. We are now in a position to solve a larger class of problems. We will expand on the simple problems we studied in the preceding chapters as a means to motivate this chapter. We will then do the essentials of tensor calculus required to derive the equations of fluid motion. The equations of motion will be derived in vector form so as to be independent of the particular coordinate system. Taken along with the tensor calculus you should be able to specialise these equations to any particular coordinate system.

5.1. Laplace Equation Revisited

We solved the Laplace equation on a unit square. How would we handle the problem domain shown in Figure 5.1? Again, we are given boundary conditions on the four sides of the quadrilateral. If we tried to solve it using a Cartesian mesh as we did before, we would get something that looks like the domain and mesh shown in Figure 5.2. Everything looks fine till you look at the top edge of the trapezium. Excepting two points, there are no other grid points on the boundary. How then can we apply the boundary condition? We could forcibly insert mesh points at the intersection of the grid lines the boundary. This would lead to unequal distances between the mesh points in our discretisation. Those points then have to be dealt with as a special case. The other possibility is to consider every grid point as being separated from neighbouring grid points by unequal distances as shown in figure 5.3. It is likely that we will have to treat some points as special points. We have come to expect that we will treat the boundary points differently from the rest of the grid points anyway. After all, look at what we did in the other problems that we have seen, especially the one-dimensional Euler equations. Since we are trying to motivate tensor calculus, our interest lies in a third possibility. That is to generate a non-Cartesian mesh. One such mesh is shown in Figure 5.4. If we look at the mesh shown in the figure, we see that the mesh lines conform to the boundary of the domain. Imagine in your mind that you have a rubber sheet the shape of this trapezium. You could stretch the sheet so that the stretched sheet looked like a
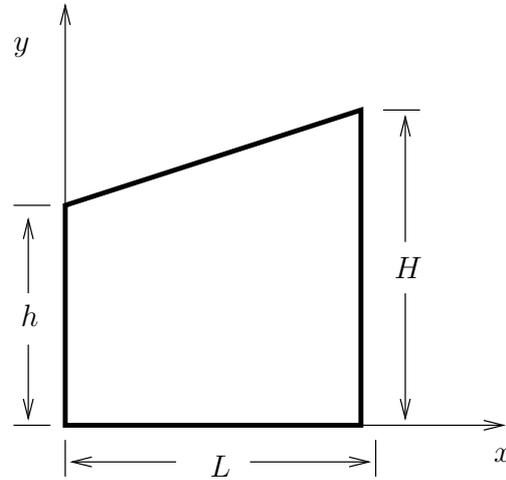
FIGURE 5.1. A trapezoidal domain on which Laplace equation is to be solved
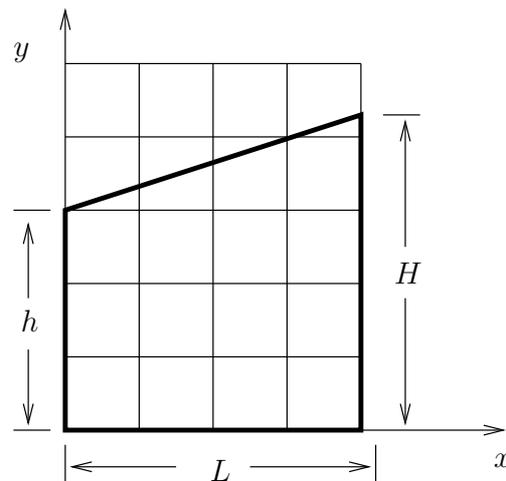
FIGURE 5.2. Trapezoidal domain with an underlying Cartesian mesh.

square. The mesh lines would coincide with the Cartesian grid lines. This tells us that we need to perform a transformation of our coordinates so that we are back to solving our problem on a Cartesian mesh.

If our coordinates in the stretched sheet are (ξ, η), the mesh lines seen in Figure 5.4 would be constant ξ-lines and constant η-lines. The Figure 5.4 is drawn in the $x - y$ plane. Clearly, what we need is a transformation going from one coordinate system to another. Say,

$$(5.1.1) \qquad \xi = \xi(x, y),$$
$$(5.1.2) \qquad \eta = \eta(x, y),$$
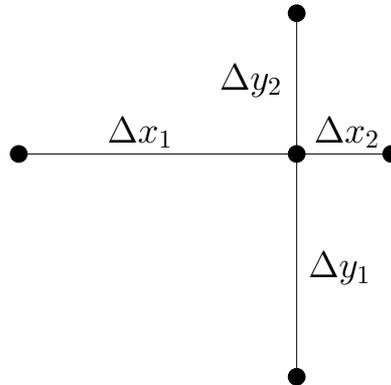
FIGURE 5.3. A grid point with neighbouring points placed at uneven distances.
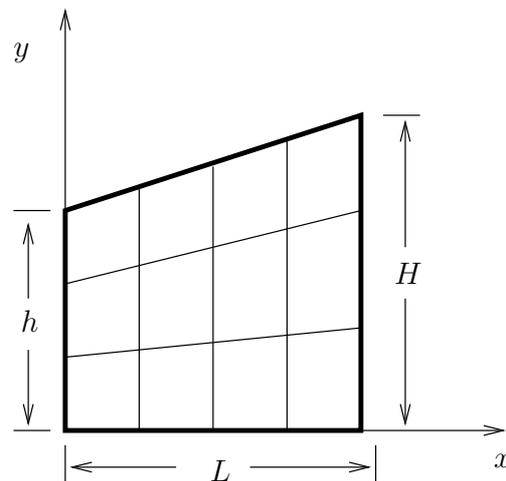
FIGURE 5.4. A non-Cartesian mesh in a Trapezoidal domain. The sides of the quadrilateral are mesh lines.

and the corresponding reverse relationship

$$(5.1.3) \qquad x = x(\xi, \eta),$$
$$(5.1.4) \qquad y = y(\xi, \eta).$$

We are in a position where, given the solution at a point in one coordinate system, we can provide the solution at the corresponding point in the other coordinate system. Let us step back for a minute to see where we are.

We have Laplace equation given to us in a trapezoidal domain in the $x - y$ coordinate system. A little stretch will give us a square in the $\xi - \eta$ coordinate system, but what happens to Laplace's equation in the $\xi - \eta$ plane? We use the coordinate transformation given by equations (5.1.1) to (5.1.3) along with chain rule to transform the derivatives. For example, here are the first derivatives in x

and y.

$$(5.1.5) \qquad \frac{\partial}{\partial x} = \underbrace{\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi}}_{A} + \underbrace{\frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta}}_{B}$$

$$(5.1.6) \qquad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}$$

Since, the transformation is known, we can determine the partial derivative on the right hand side of equation (5.1.5). How do we use the expression given by equation (5.1.5)? We can take the corresponding partial derivative of ϕ. On doing this, we get

$$(5.1.7) \qquad \frac{\partial \phi}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial \phi}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial \phi}{\partial \eta}$$

$$(5.1.8) \qquad \frac{\partial \phi}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial \phi}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial \phi}{\partial \eta}$$

So far it looks manageable. Since we want to solve Laplace's equation we now look at the second derivatives. The second x derivative is

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} \left\{ \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \right\} = \underbrace{\frac{\partial^2 \xi}{\partial x^2} \frac{\partial}{\partial \xi}}_{A_1} + \underbrace{\left(\frac{\partial \xi}{\partial x} \right)^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} \frac{\partial^2}{\partial \xi \partial \eta}}_{A_2}$$

$$(5.1.9)$$

$$+ \underbrace{\frac{\partial^2 \eta}{\partial x^2} \frac{\partial}{\partial \eta}}_{B_1} + \underbrace{\frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} \frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x} \right)^2 \frac{\partial^2}{\partial \eta^2}}_{B_2}$$

This is a little messy. To make sure we understand this clearly, the term A in equation (5.1.5) results in the terms identified as A_1 and A_2 in equation (5.1.9). The same is true of the terms marked B in the two equations. A_1 and A_2 are a consequence of applying product rule. The two terms in A_2 emerge from applying equation (5.1.5) to obtain the derivative of the $\partial/\partial \xi$ term with respect to x. In a similar fashion we can write the second derivative with respect y as

$$(5.1.10) \qquad \begin{aligned} \frac{\partial^2}{\partial y^2} &= \frac{\partial^2 \xi}{\partial y^2} \frac{\partial}{\partial \xi} + \left(\frac{\partial \xi}{\partial y} \right)^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \frac{\partial^2}{\partial \xi \partial \eta} \\ &+ \frac{\partial^2 \eta}{\partial y^2} \frac{\partial}{\partial \eta} + \frac{\partial \eta}{\partial y} \frac{\partial \xi}{\partial y} \frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y} \right)^2 \frac{\partial^2}{\partial \eta^2} \end{aligned}$$

Then the transformed Laplace equation can be written as

$$(5.1.11) \qquad \begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \left(\xi_x^2 + \xi_y^2 \right) \frac{\partial^2 \phi}{\partial \xi^2} + 2 \left(\xi_x \eta_x + \xi_y \eta_y \right) \frac{\partial^2 \phi}{\partial \xi \partial \eta} + \left(\eta_x^2 + \eta_y^2 \right) \frac{\partial^2 \phi}{\partial \eta^2} \\ &+ \left(\xi_{xx} + \xi_{yy} \right) \frac{\partial \phi}{\partial \xi} + \left(\eta_{xx} + \eta_{yy} \right) \frac{\partial \phi}{\partial \eta} = 0 \end{aligned}$$

To keep things more compact, we decide to use the notation that the subscript indicates differentiation with respect to that parameter. So,

$$(5.1.12) \qquad \xi_x = \frac{\partial \xi}{\partial x}$$

Using this notation uniformly, the Laplace equation in the $\xi - \eta$ plane is given by

$$(5.1.13) \qquad \begin{aligned} \left(\xi_x^2 + \xi_y^2\right) \phi_{\xi\xi} + 2\left(\xi_x \eta_x + \xi_y \eta_y\right) \phi_{\xi\eta} + \left(\eta_x^2 + \eta_y^2\right) \phi_{\eta\eta} \\ + \left(\xi_{xx} + \xi_{yy}\right) \phi_\xi + \left(\eta_{xx} + \eta_{yy}\right) \phi_\eta = 0 \end{aligned}$$

The domain for the problem has become easier, the equation does not quite fit in one line! Also, it is not in quite the right form. The coefficients are still expressed in the x, y coordinate system. We make the following observations and see if we can clear the air a bit.

- We want to solve problems that involve complicated domains. There may be many methods to handle complicated problems, performing transformation of coordinates is definitely one way to do it.
- We do not want to have to re-derive our governing equation in every new coordinate system that we encounter. We need a general frame work in which we can derive our equations.
- The introduction of the subscript notation gave some relief in handling the equation. So, the proper choice of notation is going to make life easier for us. Further, we can do more complex things with the effort that we are currently expending.
- We observe that the only difference between equation (5.1.9) and (5.1.10) is the replacement of x with y. Again, we need the notation that will help us to abstract these kinds of patterns out, so that we do not have to repeat the derivation for each coordinate.
- We want to solve problems in three dimensions and not just one and two dimensions. If we are going to perform transformations in three dimensions, we need to have some minimal understanding of geometry in three dimensions.

We will address the last point here by looking at a little differential geometry. Coordinate lines in three dimensions are curves in three dimensions and we will try to get a handle on them. A region of interest in three dimensions will be a volume and it is defined using surfaces. We will take a brief look at surfaces. Tensor calculus is a tool to address the rest of the issues raised in our list of observations. We will do a little tensor calculus and some geometry.

As further motivation as to why one needs tensor calculus, consider the following conundrum. If you have learnt only calculus, this is for you to puzzle over to show you there must be life beyond calculus. Consider a potential flow in two dimensions. The velocity can be written in component form as (u, v) in Cartesian coordinates. If we were to transform the velocity to some other coordinates (ξ, η) we get

$$(5.1.14) \qquad u = \frac{dx}{dt} = x_t = x_\xi \xi_t + x_\eta \eta_t = x_\xi U + x_\eta V$$

$$(5.1.15) \qquad v = \frac{dy}{dt} = y_t = y_\xi \xi_t + y_\eta \eta_t = y_\xi U + y_\eta V$$

Where (U, V) are the velocities in the $\xi - \eta$ coordinates. The matrix representation of this transformation equation is

(5.1.16)
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

We also have from the definition of the potential

(5.1.17)
$$u = \frac{\partial \phi}{\partial x} = \phi_x = \phi_\xi \xi_x + \phi_\eta \eta_x = \xi_x U + \eta_x V$$

(5.1.18)
$$v = \frac{\partial \phi}{\partial y} = \phi_y = \phi_\xi \xi_y + \phi_\eta \eta_y = \xi_y U + \eta_y V$$

Which has a representation

(5.1.19)
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

These two equations contradict each other and are **wrong**

Why are these equations, (5.1.16) and (5.1.19), different? How can the u and v transform in two different ways? One immediate conclusion that the equations are wrong. We should be able to figure out what is wrong with these equations since there are basically three terms involved. The left hand side of these two equations are clearly fine since they are the quantities with which we start and are a given. The chain rule part follows from calculus. That procedure looked right. That leaves the U and V and of course, the = symbol. We want the equation relating velocities in the two coordinate systems. That means there is a problem with the assumption that the U and V in equation (5.1.16) are the same as the U and V in equation (5.1.19). So, there may be two different kinds of U and V. To clear up these issues **study tensor calculus.**[**You93**][**Ari89**][**SS82**]! We will do a very quick overview here.

5.2. Tensor Calculus

Very often, we assume that a vector $\vec{V}$ can be written in terms of a global basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ as follows

(5.2.1)
$$\vec{V} = v^1 \hat{e}_1 + v^2 \hat{e}_2 + v^3 \hat{e}_3 = \sum_{i=0}^{3} v^i \hat{e}_i$$

We will see what we mean by a global basis as we go along. For now, do not confuse the superscript on v with exponentiation. We deliberately chose superscripts and subscripts since we anticipate that we are going to encounter two different kinds of entities. We will see that superscripted entities are said to be contravariant and subscripted entities are covariant. So, v^1 may be different from v_1. We will see what this means as we go along. If we agree that any time the index is repeated it implies a summation, we can simply write

(5.2.2)
$$\vec{V} = v^i \hat{e}_i$$

Now, THAT is compact. It is called Einstein's summation convention. It only gets better. By itself, the equation does not even restrict us to three dimensions. It is our assumption that we use three dimensions. In this book, we will restrict ourselves to two / three dimensions. You should note that

$$(5.2.3) \qquad \vec{V} = v^i \hat{e}_i = v^k \hat{e}_k$$

Since there is a summation over the index, the index itself does not survive the summation operation. The choice of the index is left to us. It is called a **dummy index**.
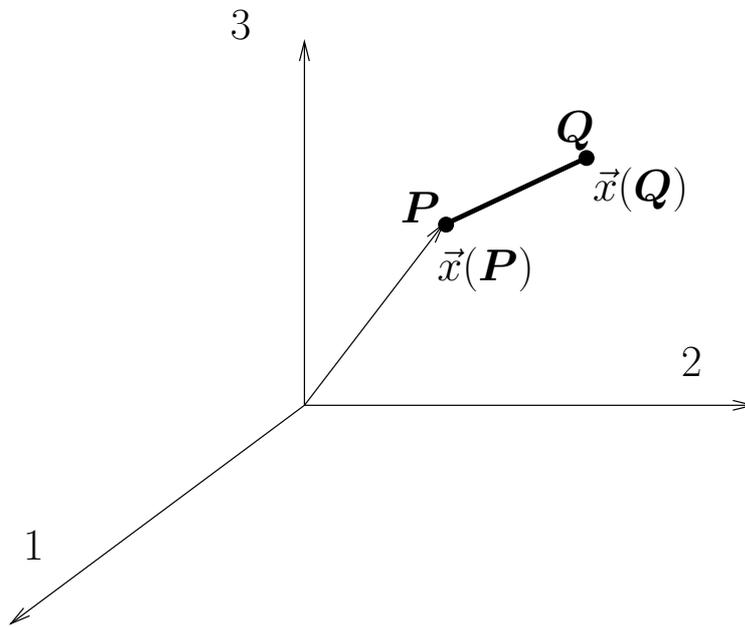
FIGURE 5.5. A Cartesian coordinate system used to locate the point P and Q. $\vec{x}(P)$ gives the position vector of P in the Cartesian coordinate system. PQ forms a differential element.

We now define the notation with respect to coordinate systems. Consider Figure 5.5. It indicates a differential line element with two points P and Q at each end of the element. We define $\vec{x}(.)$ as a coordinate function which returns the coordinates of a point in the Cartesian coordinate system. If we had another coordinate system overlayed on the same region, the point P will have the corresponding coordinates $\vec{\xi}(P)$ in that coordinate system. The coordinate function is simple to imagine if we look at it component-wise.

$$(5.2.4) \qquad \vec{x}(P) = x^1(P)\hat{e}_1 + x^2(P)\hat{e}_2 + x^3(P)\hat{e}_3$$

Since we are dealing with Cartesian coordinates, x^i and x_i are the same. we have already seen that if $\vec{P}$ is the position vector for P then

$$(5.2.5) \qquad x_i(P) = \vec{P} \cdot \hat{e}_i$$

Consider the problem of coordinate transformations in two dimensions. Let us restrict ourselves for the sake of this discussion to rotations. We take our standard $x - y$ coordinate and rotate through an angle θ to get the $\xi - \eta$ coordinates. The basis vectors in $x - y$ are $\vec{e}_1$ and $\vec{e}_2$. The basis vectors in the $\xi - \eta$ coordinates are $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$. You can check that the basis vectors are related as follows:

$$(5.2.6) \qquad \begin{pmatrix} \vec{\epsilon}_1 \\ \vec{\epsilon}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

We see that by using indices we can simply represent this as

$$(5.2.7) \qquad \vec{\epsilon}_i = A_i^j \vec{e}_j$$

Now, a vector $\vec{s}$ can be represented in the $x - y$ and the $\xi - \eta$ coordinate systems as

$$(5.2.8) \qquad \vec{s} = s^i \vec{e}_i = \psi^i \vec{\epsilon}_i$$

Substituting for $\vec{\epsilon}_i$ from equation (5.2.7) we get

$$(5.2.9) \qquad \vec{s} = s^i \vec{e}_i = s^j \vec{e}_j = \psi^i \vec{\epsilon}_i = \psi^i A_i^j \vec{e}_j$$

where i and j are dummy indices. Even though they are dummy indices, by the proper choice of these dummy indices here we can conclude that

$$(5.2.10) \qquad s^j = \psi^i A_i^j = A_i^j \psi^i$$

Compare equations (5.2.7) and (5.2.10). The unit vectors transform one way, the components transform the opposite [or contra] way. We see that they too show the same behaviour we saw with the velocity potential. Vectors that transform like each other are covariant with each other. Vectors that transform the opposite way are contravariant to each other. This is too broad a scenario for us. We will stick with something simpler. **Covariant** entities will be subscripted. **Contravariant** entities will be superscripted.

An example where this will be obvious to you is the case of the rotation of the Cartesian coordinate system. Again, we restrict ourselves to two dimensions. If you rotate the standard Cartesian coordinate system counter-clockwise, you see that the coordinate lines and the unit vectors (as expected) rotate in the same direction. They are covariant. The actual coordinate values do not change in the same fashion. In fact, the new values corresponding to a position vector look as though the coordinate system was fixed and that the position vector was rotated in a clockwise sense (contra or opposite to the original coordinate rotation). These two rotations are in fact of equal magnitude and opposite in sense. They are, indeed, inverses of each other. We will investigate covariant and contravariant quantities more as we go along. Right now, we have assumed that we have a position vector. Let us take a closer look at this.

We have made one assumption so far that the basis vector is global. We used the term global basis in the beginning of this section. What do we mean by a **global basis**? We want the basis to be the same, that is constant, at every point. Such a set of basis vectors is also said to be **homogeneous**. For example, the basis vectors in the standard Cartesian coordinate system do not depend on the (x, y) coordinates of a point. Consider the trapezium in Figure (5.7) We see that the tangent to the $\eta = $ constant coordinate lines change with η. In general, the basis vectors change from point to point. We do not have a global basis. Also, consider
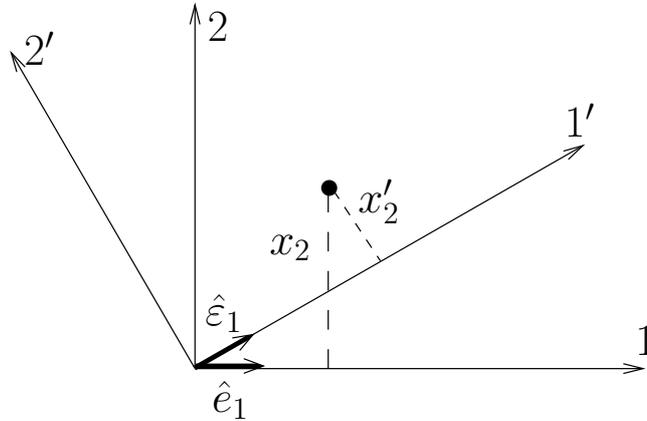
FIGURE 5.6. The basis vectors rotate with the coordinates axes (only $\hat{e}_1$ and $\vec{\varepsilon}_1$ are shown). The coordinates of the point in the new system are as though the point had moved clockwise and the coordinate system was fixed. That is $x_2' < x_2$ in this particular case.
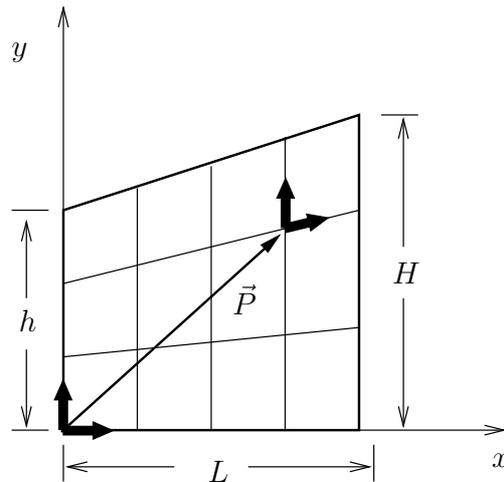
FIGURE 5.7. The basis vectors at the origin and the basis vectors at some other point are clearly not the same. The position vector, $\vec{P}$ is geometrically very easy to draw. It cannot be written simply as a linear combination of some basis vector in the coordinate system shown here.

the standard polar coordinate system (see Figure 5.8). The usual symbols for the basis vectors are $\hat{e}_\theta$ and $\hat{e}_r$. Both of these vectors depend on θ. Again, for the familiar and useful polar coordinate system, we do not have a global basis. That is the basis vectors are not constant. They are not homogeneous. They are not homogeneous. In fact, in the case of polar coordinates we have as the position vector at any point $\vec{P} = r\hat{e}_r$. Does

the position vector not depend of θ at all? The fact of the matter is that the $\hat{e}_r$ depends on θ, as the basis is not homogeneous. Fortunately, $\hat{e}_r$ and $\hat{e}_\theta$ depend only on θ. So, we are still able to write $\vec{P} = r\hat{e}_r$.
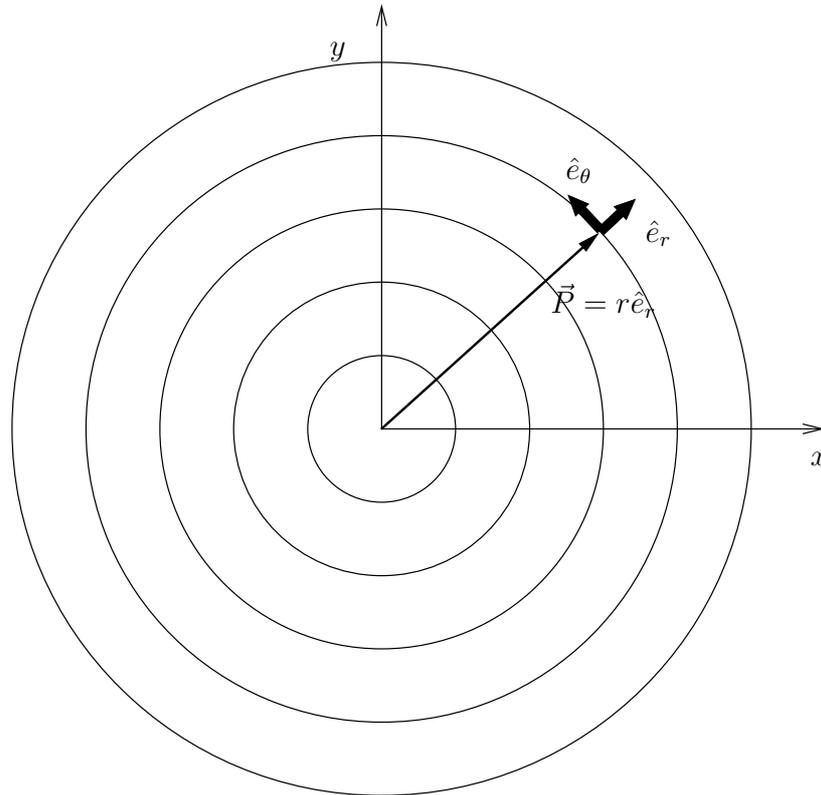
FIGURE 5.8. The position vector in polar coordinates is given by $\vec{P} = r\hat{e}_r$. At first glance it seems as though there is no θ dependence. However $\hat{e}_r$ is a function of θ as is $\hat{e}_\theta$. Only the constant r coordinate lines are shown.

Another example of a coordinate system with which you are familiar is used doing log-log plots (see Figure 5.9). In this case, the basis vectors seem to be oriented in the same fashion. However, the length of the vector seems to change. It is clear that the notion of distance between points is an issue here.

Looking at these examples, we realise that we need to spend a little time trying to understand generalised coordinate systems. Let's just consider one coordinate line in some generalised coordinate system. We will see that in three dimensions, it is a space curve. Let us first look at space curves and some of their properties. We are especially interested in the tangent to these curves since the tangent to the coordinate line is a part of our basis.

Consider the coordinate line shown in Figure 5.10. The curve is determined by a function $\vec{\alpha}(\xi^1)$. It is a coordinate line in a coordinate system labelled $\vec{\xi}$ which has three components (ξ^1, ξ^2, ξ^3). The figure shows the coordinate line corresponding
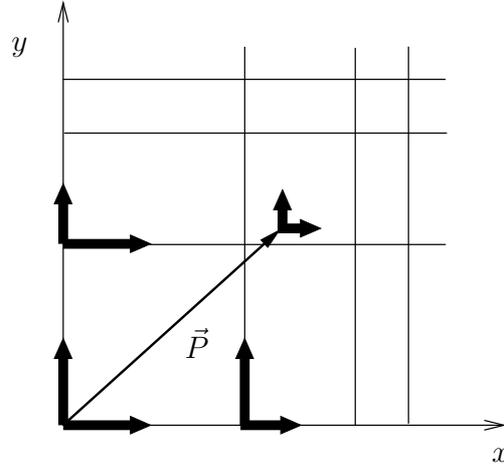
FIGURE 5.9. A "log-log" coordinate system. We have a rectangular coordinate system, however the unit vectors are still a function of position making it impossible to write the position vector drawn $\vec{P}$ as a linear combination of the basis vectors.

to $\xi^2 =$constant, and $\xi^3 =$constant. To belabour the point, it is the ξ^1 coordinate line since it is parametrised on ξ^1. The tangent to this line is given by

$$(5.2.11) \qquad \vec{\varepsilon}_1 = \frac{d\vec{\alpha}(\xi^1)}{d\xi^1}$$

In fact, for any of the coordinate lines of ξ^i we have for the corresponding $\vec{\alpha}_i$

$$(5.2.12) \qquad \vec{\varepsilon}_i = \frac{d\vec{\alpha}_i}{d\xi^i}, \quad \text{no summation on } i$$

This basis vector is called the covariant basis vector. We note the following

- In equation (5.2.12), though the subscripts are repeated, there is no summation implied over i. The subscript on the $\vec{\alpha}$ is there to conveniently indicate three generic coordinate functions.
- Some new tensor notation convention: (see equation (5.2.12)) the superscript of the derivative on the on the right hand side becomes a subscript on the left.

We consider an example to understand this process better. Let us take a look at polar coordinates in two dimensions (ξ^1, ξ^2). A $\xi^2 = \theta$ coordinate line corresponds to a $\xi^1 = r =$constant line. For the given r, the curve is parametrised as

$$(5.2.13) \qquad \vec{\alpha}(\theta) = \vec{\alpha}(\xi^2) = r\cos(\theta)\hat{\imath} + r\sin(\theta)\hat{\jmath} = \xi^1\cos(\xi^2)\hat{e}_1 + \xi^1\sin(\xi^2)\hat{e}_2$$

As was seen earlier, $\hat{e}_1$ and $\hat{e}_2$ are the standard basis vectors in the Cartesian coordinate system. You may be used to calling them $\hat{\imath}$ and $\hat{\jmath}$. The tangent to this curve is $\vec{\varepsilon}_2$.

$$(5.2.14) \qquad \vec{\varepsilon}_2 = -r\sin(\theta)\hat{e}_1 + r\cos(\theta)\hat{e}_2 = -\xi^1\sin(\xi^2)\hat{e}_1 + \xi^1\cos(\xi^2)\hat{e}_2$$

We note two points here.
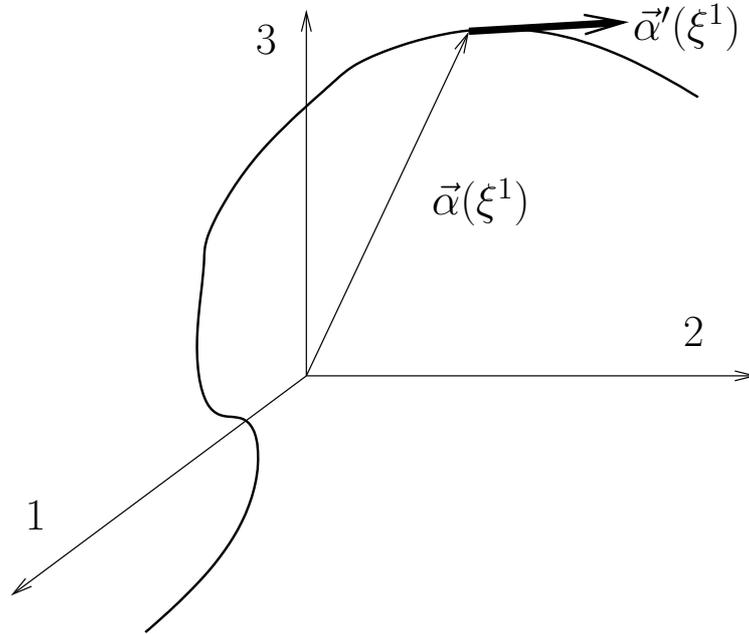
FIGURE 5.10. A coordinate line belonging to a three dimensional generalised coordinate system. This line is shown embedded in our usual Cartesian coordinate system. $\vec{\alpha}$ is shown as a function of ξ^1 alone as the other two, ξ^2 and ξ^3 are held constant to obtain this line. The local tangent vector is one of the basis vectors for the generalised coordinates

- $\vec{\varepsilon}_2$ is not a unit vector. If you normalise it, you get the "physical" basis vector $\vec{\varepsilon}_\theta$.
- $\hat{e}_1$ and $\hat{e}_2$ are not functions of ξ^2. That is the reason why we only have two terms on the right hand side of equation 5.2.14. Otherwise we would have had more derivative terms due to the application of product rule.

How about the other coordinate line corresponding to $\theta = \xi^2 =$ constant. The equation of such a line is given by

$$(5.2.15) \qquad \vec{\alpha}(r) = \vec{\alpha}(\xi^1) = \xi^1 \cos(\xi^2)\hat{e}_1 + \xi^1 \sin(\xi^2)\hat{e}_2, \quad \xi^2 = \text{constant}$$

For constant $\xi^2 = \theta$, this will correspond to a radial line. The tangent vector to this line is given by

$$(5.2.16) \qquad \vec{\varepsilon}_1 = \frac{\partial \vec{\alpha}}{\partial \xi^1} = \cos(\xi^2)\hat{e}_1 + \sin(\xi^2)\hat{e}_2$$

This in fact turns out to be a unit vector and is the same as $\vec{\varepsilon}_r$.

We can learn something from the study of the polar coordinate system. Why does $\vec{\varepsilon}_2$ depend on ξ^1? ξ^2 is the angle measured from the x-axis. The angle ξ^2 is measured in radians which is the arc length at some radius that subtends ξ^2 at the centre nondimensionalised by that radius. Naturally, when using ξ^2 as a coordinate, the corresponding arc length depends on ξ^1.
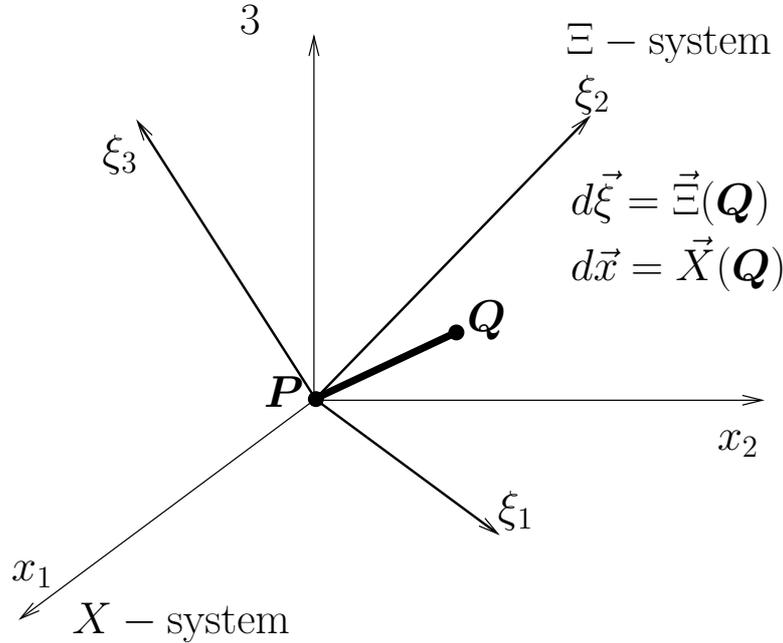
FIGURE 5.11. The origin of our Cartesian coordinate system is moved to the point $\boldsymbol{P}$. The differential element $\boldsymbol{PQ}$ is now represented in terms of the translated coordinate system and similar system of the generalised coordinates.

Let's pause and take stock of what we have and where we are. We have seen that there are coordinate systems where the basis vectors are not homogeneous. So, just writing a relation like equation (5.2.2), $\vec{V} = v^i \hat{e}_i$, for a position vector $\vec{V}$ may not be possible. We will start dealing only with differentials. A differential element $\boldsymbol{PQ}$ is shown in the Figure 5.11. It is represented in the X coordinate system as $d\vec{x} = dx^i \hat{e}_i$. The $\hat{e}_i$ are the basis vectors in this coordinate system. We can transform from the X coordinates to the Ξ coordinates where the basis vectors are $\vec{\varepsilon}_i$. The differential $\boldsymbol{PQ}$ can be written in the Ξ coordinates system as $d\vec{\xi} = d\xi^i \vec{\varepsilon}_i$. How are the two representations for the given differential element at a given point related? Clearly, the length of the element should not depend on our choice of the coordinate system. Or, put another way, if two people choose two different coordinate systems, the length of this particular element should work out to be the same. As we had done earlier, here are the two equations that relate the Cartesian coordinates x^i to the generalised coordinates ξ^i.

$$(5.2.17) \qquad x^i = x^i(\xi^1, \xi^2, \xi^3)$$

and

$$(5.2.18) \qquad \xi^i = \xi^i(x^1, x^2, x^3)$$

In the Cartesian coordinate system the length ds is given by

$$(5.2.19) \qquad (ds)^2 = d\vec{x} \cdot d\vec{x} = dx^i \hat{e}_i \cdot dx^j \hat{e}_j = dx^i dx^j \hat{e}_i \cdot \hat{e}_j$$

Remember that $\hat{e}_i$ are the basis vectors of a Cartesian coordinate system and are orthogonal to each other. Consequently, we can define a useful entity called the Kronecker delta as

$$(5.2.20) \qquad \delta_{ij} = \hat{e}_i \cdot \hat{e}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

With this new notation we can write

$$(5.2.21) \qquad (ds)^2 = d\vec{x} \cdot d\vec{x} = dx^i dx^j \delta_{ij} = dx^i dx_i = \sum_i (dx^i)^2$$

Following the convention we have used so far (without actually mentioning it) we see that

$$(5.2.22) \qquad dx^j \delta_{ij} = dx_i$$

That is, j is a dummy index and disappears leaving i which is a subscript. For the first time we have seen a contravariant quantity converted to a covariant quantity. If you think of matrix algebra for a minute, you will see that δ_{ij} is like an identity matrix. The components dx^i are the same as the components dx_i in a Cartesian coordinate system. Hence, equation (5.2.21) can be written as

$$(5.2.23) \qquad (ds)^2 = dx^i dx_i = \sum_i (dx_i)^2$$

The length of the element is invariant with transformation meaning the choice of our coordinates should not change the length of the element. A change to the Ξ coordinates should give us the same length for the differential element $\boldsymbol{PQ}$. The length in the Ξ coordinates is given by

$$(5.2.24) \qquad (ds)^2 = d\vec{\xi} \cdot d\vec{\xi} = d\xi^i \vec{\varepsilon}_i \cdot d\xi^j \vec{\varepsilon}_j = d\xi^i d\xi^j \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j = d\xi^i d\xi^j g_{ij} = d\xi^i d\xi_i$$

g_{ij} is called the metric. Following equation (5.2.22), we have defined $d\xi_i = g_{ij} d\xi^j$. Why did we get g_{ij} instead of δ_{ij}? We have seen in the case of the trapezium that the basis vectors need not be orthogonal to each other since the coordinate lines are not orthogonal to each other. So, the dot product of the basis vectors $\vec{\varepsilon}_i$ and $\vec{\varepsilon}_j$ gives us a g_{ij} with non-zero off-diagonal terms. It is still symmetric, though. In this case, unlike the Cartesian situation, $d\xi^i$ is different from $d\xi_i$.

We can define another set of basis vectors which are orthogonal to the covariant set as follows

$$(5.2.25) \qquad \vec{\varepsilon}_i \cdot \vec{\varepsilon}^{\,j} = \delta_i^j$$

where,

$$(5.2.26) \qquad \delta_i^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This new basis, $\vec{\varepsilon}^{\,i}$, is called the contravariant basis or a dual basis. This is demonstrated graphically in figure 5.12. This basis can be used to define a metric

$$(5.2.27) \qquad g^{ij} = \vec{\varepsilon}^{\,i} \cdot \vec{\varepsilon}^{\,j}$$
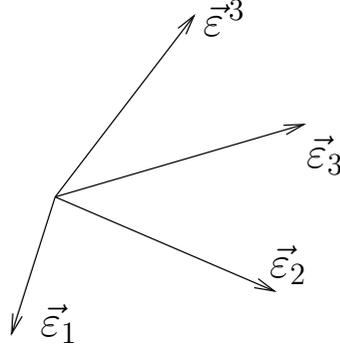
FIGURE 5.12. The covariant basis vectors $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$, and $\vec{\varepsilon}_3$ are shown. In general they may not be orthogonal to each other. $\vec{\varepsilon}^3$ is also shown. It is orthogonal to $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ and $\vec{\varepsilon}_3 \cdot \vec{\varepsilon}^3 = 1$

Now, is the definition given for $d\xi_i$ consistent with this definition of the contravariant basis? Is $d\vec{\xi} = d\xi_i \vec{\varepsilon}^i$? That is, if we take the dot product of a vector with a basis vector, do we get the corresponding component? We have,

$$(5.2.28) \qquad d\vec{\xi} = d\xi_i \vec{\varepsilon}^i \Rightarrow \quad d\vec{\xi} \cdot \vec{\varepsilon}^j = d\xi_i \underbrace{\vec{\varepsilon}^i \cdot \vec{\varepsilon}^j}_{g^{ij}} = d\xi^j,$$

and

$$(5.2.29) \qquad d\vec{\xi} = d\xi_i \vec{\varepsilon}^i \Rightarrow \quad d\vec{\xi} \cdot \vec{\varepsilon}_j = d\xi_i \vec{\varepsilon}^i \cdot \vec{\varepsilon}_j = d\xi_j,$$

and

$$(5.2.30) \qquad d\vec{\xi} = d\xi^i \vec{\varepsilon}_i \Rightarrow \quad d\vec{\xi} \cdot \vec{\varepsilon}_j = d\xi^i \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j = d\xi_j,$$

and finally,

$$(5.2.31) \qquad d\vec{\xi} = d\xi^i \vec{\varepsilon}_i \Rightarrow \quad d\vec{\xi} \cdot \vec{\varepsilon}^j = d\xi^i \vec{\varepsilon}_i \cdot \vec{\varepsilon}^j = d\xi^j,$$

So, to get the contravariant components of a tensor, dot it with the contravariant basis vectors. Likewise, to get the covariant components of a tensor, dot it with the covariant basis vectors. The effect of g_{ij} on a contravariant term is to lower the index or convert it to a covariant term. Similarly, the effect of g^{ij} on a covariant term is to raise the index or convert it to a contravariant term. So, what is $g_{ij}g^{jk}$?

$$(5.2.32) \qquad g_{ij}g^{jk} = g_i^k = \vec{\varepsilon}_i \cdot \vec{\varepsilon}^k = \delta_i^k$$

The metric tensors are inverses of each other.

At this point you really can protest: Wait a minute, where is this going? "Fascinating" as it is, how is it relevant to CFD? Look at the trapezium in Figure 5.4. Imagine that this trapezium represent a channel through which some fluid, like water, can flow. The top and bottom of the trapezium shown are solid walls. If we were solving for the potential flow through a channel with the top and bottom of the trapezium being solid walls, this tells us, we need to apply the boundary condition $\partial\phi/\partial n = 0$, where n is measured along a line that is perpendicular to the surface. Look at the top of the trapezium. A zoomed view is shown in Figure 5.13. Your coordinate line is not normal to the top surface. How do we get the derivative along the normal. You can find the derivatives along $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ and use Taylor's
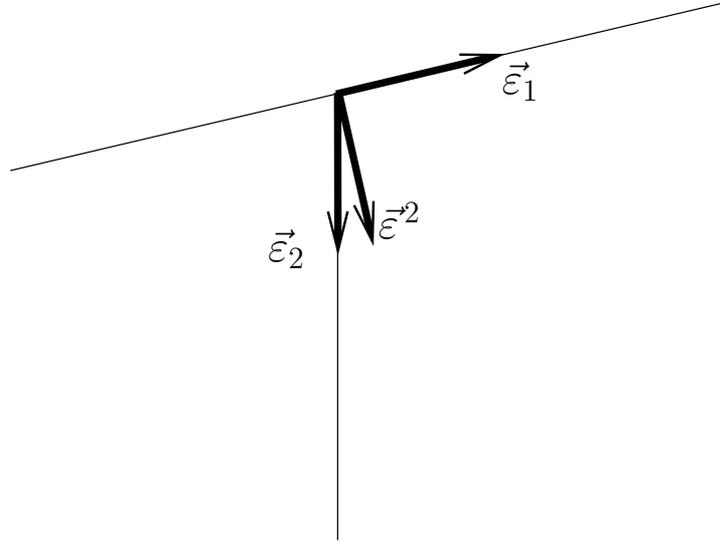
FIGURE 5.13. A zoomed view of the non-Cartesian mesh in a Trapezoidal domain shown in Figure 5.4. The two covariant basis vectors and one contravariant basis vector are shown.

series in two dimensions to get the normal derivative. You will find that you are just reinventing everything we have done so far. What you want is the contravariant basis vector and not the covariant basis vector. Why? This is because the covariant basis vector is along the coordinate line and the contravariant one is perpendicular to it. The top of the trapezium is a coordinate line. The contravariant basis vector is perpendicular to it, which is what we want. We do need this stuff, so let's soldier on. First, an assignment.

Assignment 5.1

(1) Expand the following using the summation convention assuming that we are working in three dimensions.
   (a) $a^i b^j \delta_{ij}$,   (b) δ^j_j,   (c) $\delta^j_i \delta^i_j$,   (d) $\delta^i_i \delta^j_j$
(2) Repeat the above problem assuming we are dealing with tensors in two space dimensions.
(3) Find the covariant and contravariant bases vectors and the corresponding metric tensors for the following coordinate systems. x^i are the Cartesian coordinates.
   (a) Cylindrical coordinates. $\xi^1 = r$, $\xi^2 = \theta$, and $\xi^3 = z$ in conventional notation.
   $x^1 = \xi^1 \cos \xi^2$ $x^2 = \xi^1 \sin \xi^2$ $x^3 = \xi^3$.
   (b) Spherical coordinates. $\xi^1 = R$, $\xi^2 = \theta$, and $\xi^3 = \phi$ in conventional notation.
   $x^1 = \xi^1 \sin \xi^2 \cos \xi^3$ $x^2 = \xi^1 \sin \xi^2 \sin \xi^3$ $x^3 = \xi^1 \cos \xi^2$
   (c) Parabolic cylindrical coordinates.
   $x^1 = \frac{1}{2} \left\{ \left(\xi^1 \right)^2 - \left(\xi^2 \right)^2 \right\}$ $x^2 = \xi^1 \xi^2$ $x^3 = \xi^3$.

(4) Compute the covariant and contravariant velocity components in the above coordinate systems.

You have seen in multivariate calculus that given a smooth function ϕ, in a region of interest, we can find the differential $d\phi$ as

$$(5.2.33) \qquad d\phi = \frac{\partial \phi}{\partial \xi^i} d\xi^i$$

Now, we also know that this is a directional derivative and can be written as

$$(5.2.34) \qquad d\phi = \nabla\phi \cdot d\vec{\xi} = \frac{\partial \phi}{\partial \xi^i} d\xi^i$$

where,

$$(5.2.35) \qquad \nabla = \vec{\varepsilon}^{\,j} \frac{\partial}{\partial \xi^j}, \quad d\vec{\xi} = \vec{\varepsilon}_i d\xi^i$$

We managed to define the gradient operator ∇. What happens when we take the gradient of a vector? How about the divergence? We first write the gradients of a scalar function and a vector function as

$$(5.2.36) \qquad \nabla\phi = \vec{\varepsilon}^{\,j} \frac{\partial \phi}{\partial \xi^j}$$

$$(5.2.37) \qquad \nabla\vec{V} = \vec{\varepsilon}^{\,j} \frac{\partial \vec{V}}{\partial \xi^j}$$

If we look carefully at the two equation above, we see that equation (5.2.37) is different. It involves, due to the use of product rule, the derivatives of the basis vectors. In fact, equation (5.2.37) can written as

$$(5.2.38) \qquad \nabla\vec{V} = \vec{\varepsilon}^{\,j} \frac{\partial \vec{V}}{\partial \xi^j} = \vec{\varepsilon}^{\,j} \left\{ \frac{\partial v^i}{\partial \xi^j} \vec{\varepsilon}_i + v^i \frac{\partial \vec{\varepsilon}_i}{\partial \xi^j} \right\}$$

So, what is the nature of the derivative of the basis vector? For one thing, from the definition of the covariant basis in equation (5.2.12) we have

$$(5.2.39) \qquad \frac{\partial \vec{\varepsilon}_i}{\partial \xi^{\,j}} = \frac{\partial^2 \vec{\alpha}}{\partial \xi^j \partial \xi^i} = \frac{\partial \vec{\varepsilon}_j}{\partial \xi^i}$$

We have dispensed with the subscript on $\vec{\alpha}$ so as not to create more confusion. We will use the correct $\vec{\alpha}$ corresponding to the coordinate line. We can see from equation (5.2.39) that its component representation is going to be symmetric in the two indices i and j. As we have already seen in equation (5.2.28), to find the contravariant components of this entity we can dot it with $\vec{\varepsilon}^{\,k}$ to get

$$(5.2.40) \qquad \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = \vec{\varepsilon}^{\,k} \cdot \frac{\partial \vec{\varepsilon}_i}{\partial \xi^j}$$

$\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}$ is called a Christoffel symbol of the second kind. We took the dot product with $\vec{\varepsilon}^{\,k}$ so that equation (5.2.38) can be rewritten as

$$(5.2.41) \qquad \nabla\vec{V} = \vec{\varepsilon}^{\,j} \left\{ \frac{\partial v^i}{\partial \xi^j} \vec{\varepsilon}_i + v^i \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \vec{\varepsilon}_k \right\}$$

Since i and k are dummy indices (meaning we are going to sum over their values) we swap them for a more convenient expression

$$(5.2.42) \qquad \nabla \vec{V} = \vec{\varepsilon}^{\,j} \left\{ \frac{\partial v^i}{\partial \xi^j} \vec{\varepsilon}_i + v^k \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \vec{\varepsilon}_i \right\}$$

This allows us to write

$$(5.2.43) \qquad \frac{\partial \vec{V}}{\partial \xi^j} = \left\{ \frac{\partial v^i}{\partial \xi^j} + v^k \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \right\} \vec{\varepsilon}_i$$

In pure component form this is written as

$$(5.2.44) \qquad v^i_{;j} = \frac{\partial v^i}{\partial \xi^j} + v^k \left\{ \begin{matrix} i \\ kj \end{matrix} \right\}$$

This is called the covariant derivative of the contravariant vector v^i. Staying with our compact notation, the covariant derivative is indicated by the semi-colon in the subscript. This is so that we do not confuse it with the plain derivative $\partial v^i / \partial \xi^j$.

So, if we have Christoffel symbols of the second kind do we have any other kind? Yes, there is a Christoffel symbol of the first kind. It is written as $[ij, k]$ and it is given by

$$(5.2.45) \qquad [ij, k] = \left\{ \begin{matrix} l \\ ij \end{matrix} \right\} g_{lk} = \frac{\partial \vec{\varepsilon}_i}{\partial \xi^j} \cdot \vec{\varepsilon}^{\,l} g_{lk} = \frac{\partial \vec{\varepsilon}_i}{\partial \xi^j} \cdot \vec{\varepsilon}_k$$

The Christoffel symbols of the first kind can be directly obtained as

$$(5.2.46) \qquad [ij, k] = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ki}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^k} \right)$$

This can be verified by substituting for the definition of the metric tensor. The peculiar notation with brackets and braces is used for the Christoffel symbols (and they are called symbols) because, it turns out that they are not tensors. That is, though they have indices, they do not transform the way tensors do when going from one coordinate system to another. We are not going to show this here. However, we should not be surprised that they are not tensors as the Christoffel symbols encapsulate the relationship of the two coordinate systems and would necessarily depend on the coordinates.

The divergence of $\vec{V}$ is defined as the trace of the gradient of $\vec{V}$. That is

$$(5.2.47) \qquad \text{div} \vec{V} = \vec{\varepsilon}^{\,j} \cdot \left\{ \frac{\partial v^i}{\partial \xi^j} \vec{\varepsilon}_i + v^k \left\{ \begin{matrix} i \\ kj \end{matrix} \right\} \vec{\varepsilon}_i \right\}$$

Assignment 5.2

For the coordinate systems given in assignment 5.1,

(1) Find the Christoffel symbols of the first and second kind.
(2) Find the expression for the gradient of a scalar potential.
(3) Find the gradient of the velocity vector.
(4) Find the divergence of the velocity vector.
(5) Find the divergence of the gradient of the scalar potential that you just found.

In the case of the velocity potential $\vec{V} = \nabla\phi$ we get,

$$(5.2.48) \qquad \vec{V} = \vec{\varepsilon}^{\,j}\frac{\partial\phi}{\partial\xi^j} = \vec{\varepsilon}_k g^{kj}\frac{\partial\phi}{\partial\xi^j} = \vec{\varepsilon}_k g^{kj}v_j = v^k\vec{\varepsilon}_k$$

If we now take the divergence of this vector using equation (5.2.47) we get
(5.2.49)
$$\nabla^2\phi = \vec{\varepsilon}^{\,j}\cdot\left\{\frac{\partial v^i}{\partial\xi^j}\vec{\varepsilon}_i + v^k\left\{{i \atop kj}\right\}\vec{\varepsilon}_i\right\} = \vec{\varepsilon}^{\,j}\cdot\left\{\frac{\partial}{\partial\xi^j}\left(g^{il}\frac{\partial\phi}{\partial\xi^l}\right)\vec{\varepsilon}_i + v^k\left\{{i \atop kj}\right\}\vec{\varepsilon}_i\right\}$$

Completing the dot product we get

$$(5.2.50) \qquad \nabla^2\phi = \left\{\frac{\partial}{\partial\xi^i}\left(g^{il}\frac{\partial\phi}{\partial\xi^l}\right) + v^k\left\{{i \atop ki}\right\}\right\}$$

Substituting for v^k from equation (5.2.48) we get

$$(5.2.51) \qquad \nabla^2\phi = \left\{\frac{\partial}{\partial\xi^i}\left(g^{il}\frac{\partial\phi}{\partial\xi^l}\right) + g^{kl}\frac{\partial\phi}{\partial\xi^l}\left\{{i \atop ki}\right\}\right\}$$

This much tensor calculus will suffice. A more in depth study can be made using the numerous books that are available on the topic [**You93**], [**SS82**].

5.3. Equations of Fluid Motion

We have seen enough tensor calculus so that if we derive the governing equations in some generic coordinate system, we can always transform the resulting equations into any other coordinate system. In fact, as far as possible, we will derive the equations in vector form so that we can pick the component form that we feel is appropriate for us. We can conveniently use the Cartesian coordinate system for the derivation with out loss of generality.

We will first derive the equations of motion in integral form. We will do this in a general setting. Let us consider some fluid property $\mathcal{Q}$, whose property density is given by Q. For example, consider a situation in which we have added some ink to flowing water. At any given time, the mass of ink in a small elemental region of interest may be dm_{ink}. If the volume of the elemental region is $d\sigma$, then these two measures defined on that region are related through the ink density as

$$(5.3.1) \qquad dm_{\text{ink}} = \frac{dm_{\text{ink}}}{d\sigma}d\sigma = \rho_{\text{ink}}d\sigma$$

We would like to write out the balance laws for a general property, $\mathcal{Q}$. We **arbitrarily** pick a control volume. One such volume is indicated in the Figure 5.14. For the sake of simplicity, we pick a control volume that does not change in time. This control volume occupies a region of volume σ. This control volume has a surface area S. It is located as shown in the figure and is immersed in a flow field. Within this control volume, at an arbitrary point $\vec{x}$, we pick a small elemental region with volume $d\sigma$. From equation (5.3.1), the amount of the property of interest at time t, $d\mathcal{Q}(\vec{x}, t)$, in the elemental control volume is $Q(\vec{x}, t)d\sigma$. Then the total quantity contained in our control volume at any instant is

$$(5.3.2) \qquad \mathcal{Q}_\sigma(t) = \int_\sigma Q(\vec{x}, t)d\sigma$$
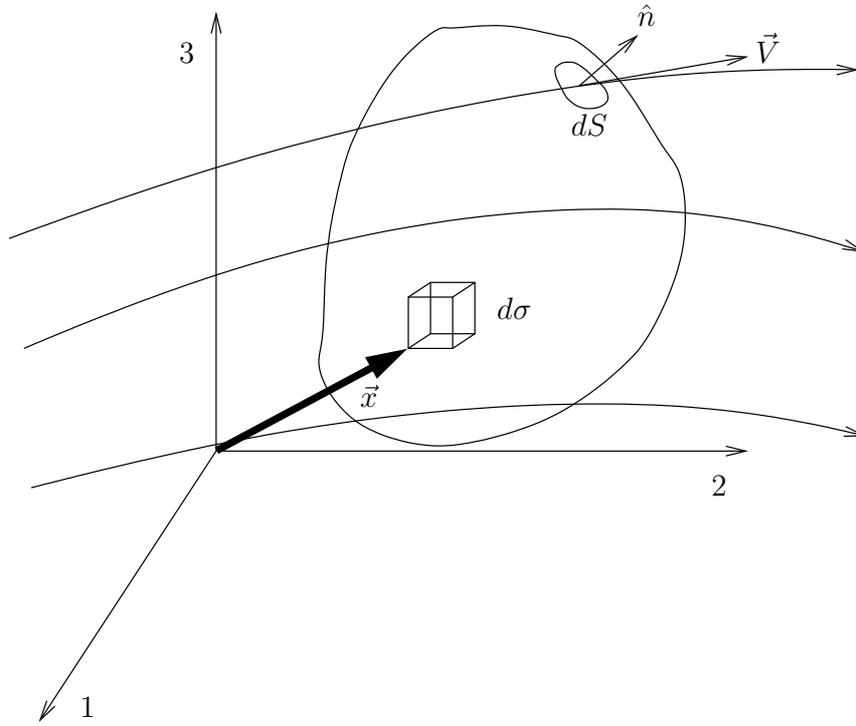
FIGURE 5.14. An arbitrary control volume chosen in some fluid flow. An elemental area on the controls surface dS and and elemental volume $d\sigma$ within the control volume are also shown. Note that in most coordinate systems we may not be able to indicate a position vector $\vec{x}$.

The time rate of change of this quantity is

$$(5.3.3) \qquad \frac{d\mathcal{Q}_\sigma}{dt} = \frac{d}{dt}\int_\sigma Q(\vec{x},t)d\sigma$$

Then we ask ourselves the question, why is there a rate of change? There is change because the property $\mathcal{Q}$ is carried / transported in and out of the control volume by the fluid. It is also possible, based on the nature of $\mathcal{Q}$, that it is somehow created or destroyed in the control volume. There may be many mechanisms by which $\mathcal{Q}$ can be changed into some other property. Let us now look at the transport of $\mathcal{Q}$ by the flow.

At any arbitrary point on the surface of the control volume that we have shown in Figure 5.14, we can determine the unit surface normal vector. We can pick a small elemental area dS at that point. The surface normal is perpendicular to this element. By convention, we choose to pick a surface normal that points out of the control volume. The rate at which our property $\mathcal{Q}$ flows out through this elemental

area is given by $Q\vec{V} \cdot \hat{n}dS$. The total efflux (outflow) from the control volume is

$$(5.3.4) \qquad \int_S Q\vec{V} \cdot \hat{n}dS$$

Since this is a net efflux, it would cause a decrease in the amount of $\mathcal{Q}$ contained in the control volume. So, our balance law can be written as

$$(5.3.5) \qquad \frac{d}{dt}\int_\sigma Qd\sigma = -\int_S Q\vec{V} \cdot \hat{n}dS + \text{ any other mechanism to produce } \mathcal{Q}$$

Before going on we will make the following observation. Though the control volume can be picked arbitrarily, we will make sure that it is smooth enough to have surface normals almost everywhere. Almost everywhere? If you think of a cube, we cannot define surface normals at the edges and corners. We can break up the surface integral in equation (5.3.5) into the sum of six integrals, one for each face of the cube.

5.3.1. Conservation of Mass. Let us look at an example. If the property we were considering was mass, $\mathcal{Q}_\sigma(t)$ would be the mass of fluid in our control volume at any given time. The corresponding Q would be mass density which we routinely refer to as the density, ρ. Ignoring mechanisms to create and destroy or otherwise modify mass, we see that the production terms disappear, leaving only the first term on the right hand side of equation (5.3.5). This gives us the equation for **balance of mass** as

$$(5.3.6) \qquad \frac{d}{dt}\int_\sigma \rho d\sigma = -\int_S \rho\vec{V} \cdot \hat{n}dS$$

This equation is also called the **conservation of mass** equation.

5.3.2. Conservation of Linear Momentum. On the other hand, if we consider the property $\mathcal{Q}$ to be momentum, the property density Q turns out to be $\rho\vec{V}$, which is the momentum density. In this case, we know that the total momentum in the control volume can also be changed by applying forces. For the sake of this discussion, forces come in two flavours. There are those that correspond to action across a distance, these forces are often called body forces. The others that depend on proximity are called surface forces[1]. We can write our equation of **balance of linear momentum** as

$$(5.3.7) \qquad \frac{d}{dt}\int_\sigma \rho\vec{V} d\sigma = -\int_S \rho\vec{V}\vec{V} \cdot \hat{n}dS + \int_\sigma \vec{f} d\sigma + \int_S \vec{T}dS$$

Here, $\vec{f}(\vec{x})$ is the body force per unit volume at the point $\vec{x}$ within the control volume. $\vec{T}(\vec{x})$ is the traction force per unit area acting at some point $\vec{x}$ on the control surface. If we are willing or able to ignore the body force, we are left with the traction force to be handled. From fluid mechanics, you would have seen that we can associate at a point, a linear transformation called the stress tensor, which relates the normal to a surface element to the traction force on that element. That is

$$(5.3.8) \qquad \vec{T} = \boldsymbol{\tau} \cdot \hat{n}$$

---

[1] As with everything that we do in physics, what we mean by this really depends on length scales. We have assumed that we are dealing with a continuum and that implicitly has a bifurcation of the length scales built into it.

where, $\vec{T} = T_i \vec{\varepsilon}^i$, $\boldsymbol{\tau} = \tau_{ij} \vec{\varepsilon}^i \vec{\varepsilon}^j$, and $\hat{n} = n_k \vec{\varepsilon}^k$. This gives us the Cauchy equation in component form as

$$(5.3.9) \qquad\qquad T_i = \tau_{ij} n^j$$

The **momentum balance** equation can be written as

$$(5.3.10) \qquad \frac{d}{dt} \int_\sigma \rho \vec{V} d\sigma = - \int_S \rho \vec{V} \vec{V} \cdot \hat{n} dS + \int_S \boldsymbol{\tau} \cdot \hat{n} dS$$

Combining terms we get

$$(5.3.11) \qquad \frac{d}{dt} \int_\sigma \rho \vec{V} d\sigma = - \int_S \left\{ \rho \vec{V} \vec{V} - \boldsymbol{\tau} \right\} \cdot \hat{n} dS$$

5.3.3. Conservation of Energy. Finally, if we consider the total energy as the property of interest so that we write out the balance law for energy. Considering the form of the first two equations, we will define the total energy density as ρE_t, where E_t is the specific total energy defined as

$$(5.3.12) \qquad\qquad E_t = e + \frac{1}{2} \vec{V} \cdot \vec{V},$$

Where e is the specific internal energy defined for a perfect gas as $e = C_v T$. C_v is the specific heat at constant volume and T is the temperature measured on the Kelvin scale. We need to look at the production terms again in equation (5.3.5). The total energy in our control volume can be changed by

(1) the forces from the earlier discussion doing work on the control volume,
(2) the transfer of energy by the process of heat through radiation and conduction,
(3) the apparent creation of energy through exo-thermic or endo-thermic chemical reactions,
(4) and finally, of course, the transportation of energy across the control surface by the fluid.

We will ignore radiation and chemical reactions here. This results in the balance law

$$(5.3.13) \quad \frac{d}{dt} \int_\sigma \rho E_t d\sigma = - \int_S \rho E_t \vec{V} \cdot \hat{n} dS + \int_\sigma \vec{f} \cdot \vec{V} d\sigma + \int_S \vec{T} \cdot \vec{V} dS - \int_S \vec{q} \cdot \hat{n} dS$$

Here, $\vec{q}$ is the term quantifying heat. Again, if we are in a position to ignore body forces we get

$$(5.3.14) \qquad \frac{d}{dt} \int_\sigma \rho E_t d\sigma = - \int_S \rho E_t \vec{V} \cdot \hat{n} dS + \int_S \vec{V} \cdot \boldsymbol{\tau} \cdot \hat{n} dS - \int_S \vec{q} \cdot \hat{n} dS$$

which we conveniently rewrite incorporating the other balance laws as

$$(5.3.15) \qquad\qquad \frac{d}{dt} \int_\sigma Q d\sigma = - \int_S \vec{\mathcal{F}} \cdot \hat{n} dS$$

where we have

$$(5.3.16) \qquad Q = \left\{ \begin{array}{c} \rho \\ \rho \vec{V} \\ \rho E_t \end{array} \right\}, \quad \vec{\mathcal{F}} = \left\{ \begin{array}{c} \rho \vec{V} \\ \rho \vec{V} \vec{V} - \boldsymbol{\tau} \\ (\rho E_t) \vec{V} - \boldsymbol{\tau} \cdot \vec{V} + \vec{q} \end{array} \right\}$$

where, $\boldsymbol{\tau} \cdot \vec{V}$ is the rate at which the traction force does work on the control volume. This, gives us a consolidated statement for the balance (conservation) of mass,

linear momentum, and energy. The great thing about this equation is that it can be cast in any three dimensional coordinate system to get the component form. It is written in a coordinate free fashion. Though, it is good to admire, we finally need to solve a specific problem, so we pick a coordinate system convenient for the solution of our problem and express these equations in that coordinate system. The other problem is that as it is there is some element of ambiguity in the dot products of the form $(\boldsymbol{\tau} \cdot \vec{V}) \cdot \hat{n}$. These ambiguities are best resolved in terms of components.

$$(5.3.17) \qquad \vec{T} \cdot \vec{V} = T_i \vec{\varepsilon}^i \cdot \vec{\varepsilon}_l V^l = \tau_{ij} \vec{\varepsilon}^i (\vec{\varepsilon}^j \cdot \vec{\varepsilon}^k) n_k \cdot \vec{\varepsilon}_l V^l = \tau_{ij} n^j V^i$$

The differential form of equation (5.3.15) can be obtained by applying the theorem of Gauss to the right hand side of the equation and converting the surface integral to a volume integral.

$$(5.3.18) \qquad \int_\sigma \left\{ \frac{\partial Q}{\partial t} + \mathrm{div} \vec{\mathcal{F}} \right\} d\sigma = 0$$

The control volume is chosen arbitrarily. As a consequence, the integral needs to be zero for any σ over which we integrate. This is possible only if

$$(5.3.19) \qquad \frac{\partial Q}{\partial t} + \mathrm{div} \vec{\mathcal{F}} = 0$$

The form of equation (5.3.15) is quite general. We could add, as required, more terms to the $\vec{\mathcal{F}}$ on the right hand side. We could also add as many equations as required. If you have other properties that need to be tracked, the corresponding equations can be incorporated. However, for our purpose, these equations are quite general. We will start with a little specialisation and simplification.

We now decompose the stress tensor $\boldsymbol{\tau}$ into a spherical part and a deviatoric part. The spherical part we will assume is the same as the pressure we have in the equation of state. The deviatoric part [or the deviation from the sphere] will show up due to viscous effects. So, $\boldsymbol{\tau}$ can be written as

$$(5.3.20) \qquad \boldsymbol{\tau} = -p\mathbf{1} + \boldsymbol{\sigma}$$

$\mathbf{1}$ is the unit tensor and $\boldsymbol{\sigma}$ is the deviatoric part. Do not confuse $\boldsymbol{\sigma}$ a tensor with the control volume σ. Through thermodynamics, we have an equation of state / constitutive model for p. Typically, we use something like $p = \rho RT$, where T is the temperature in Kelvin and R is the gas constant. We need to get a similar equation of state / constitutive model for $\boldsymbol{\sigma}$. Assuming the fluid is a Navier-Stokes fluid, that is the fluid is Newtonian, isotropic and Stokes hypothesis holds we get

$$(5.3.21) \qquad \boldsymbol{\sigma} \quad = \quad -\frac{2}{3}\mu \mathrm{tr}\boldsymbol{D} + 2\mu\boldsymbol{D}, \quad \text{where}$$

$$(5.3.22) \qquad \boldsymbol{D} \quad = \quad \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \quad \text{and}$$

$$(5.3.23) \qquad \boldsymbol{L} \quad = \quad \nabla \vec{V}$$

where μ is the coefficient of viscosity and $\mathrm{tr}\boldsymbol{D}$ is the trace of $\boldsymbol{D}$. Which is the sum of the diagonals of the matrix representation of the tensor. $\boldsymbol{L}^T$ is the transpose of $\boldsymbol{L}$. Really, $\boldsymbol{D}$ is the symmetric part of the the gradient of $\vec{V}$. Since we are right now looking at inviscid flow, we can ignore the viscous terms. So, for the Euler's equation we have

$$(5.3.24) \qquad \vec{T} = -p\mathbf{1} \cdot \hat{n}$$

where, **1** is the unit tensor. The Euler's momentum conservation equation can be written as

$$(5.3.25) \qquad \frac{d}{dt} \int_\sigma \rho \vec{V} d\sigma = - \int_S \rho \vec{V} \vec{V} \cdot \hat{n} dS - \int_S p \mathbf{1} \cdot \hat{n} dS$$

Combining terms we get

$$(5.3.26) \qquad \frac{d}{dt} \int_\sigma \rho \vec{V} d\sigma = - \int_S \left\{ \rho \vec{V} \vec{V} + p \mathbf{1} \right\} \cdot \hat{n} dS$$

which we conveniently rewrite as

$$(5.3.27) \qquad \frac{d}{dt} \int_\sigma Q d\sigma = - \int_S \vec{\mathcal{F}} \cdot \hat{n} dS$$

where we have

$$(5.3.28) \qquad Q = \left\{ \begin{array}{c} \rho \\ \rho \vec{V} \\ \rho E_t \end{array} \right\}, \quad \vec{\mathcal{F}} = \left\{ \begin{array}{c} \rho \vec{V} \\ \rho \vec{V} \vec{V} + p \mathbf{1} \\ (\rho E_t + p) \vec{V} \end{array} \right\}$$

giving us a consolidated statement for the conservation [or balance] of mass, linear momentum, and energy. These equations are collectively referred to as the Euler's equation. There are, as is usual, a set of auxiliary equations to complement these equations. The constitutive model given by the equation of state is

$$(5.3.29) \qquad p = \rho RT$$

and

$$(5.3.30) \qquad E_t = e + \frac{\vec{V} \cdot \vec{V}}{2}$$

$$(5.3.31) \qquad e = C_v T$$

With these added equations we have a closed set of equations that we should be able to solve. The equations are in integral form. We can employ the theorem of Gauss on the surface integral in equation (5.3.27) and convert it to a volume integral like so

$$(5.3.32) \qquad \frac{d}{dt} \int_\sigma Q d\sigma = - \int_S \vec{\mathcal{F}} \cdot \hat{n} dS = - \int_\sigma \mathrm{div} \vec{\mathcal{F}} d\sigma$$

This gives us the following equation

$$(5.3.33) \qquad \int_\sigma \left(\frac{\partial Q}{\partial t} + \mathrm{div} \vec{\mathcal{F}} \right) d\sigma = 0$$

which is valid for all possible control volumes on which we have surface normals and can perform the necessary integration. Remember, this "particular" σ was chosen arbitrarily. We conclude that the integral can be zero for any σ only if the integrand is zero. The differential form of the Euler's equation can be written as

$$(5.3.34) \qquad \frac{\partial Q}{\partial t} + \mathrm{div} \vec{\mathcal{F}} = 0$$

If we use normal convention to write $\vec{\mathcal{F}}$ in Cartesian coordinates as

$$(5.3.35) \qquad \vec{\mathcal{F}} = E \hat{\imath} + F \hat{\jmath} + G \hat{k}$$

our governing equation in Cartesian coordinates then becomes

$$(5.3.36) \qquad \frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0$$

Clearly, given any other basis vector, metrics, Christoffel symbols, we can write the governing equations in the corresponding coordinate system.

Assignment 5.3

(1) Given the concentration of ink at any point in a flow field is given by c_i, derive the conservation equation in integral form for ink. The diffusivity of ink is D_i.
(2) From the integral from in the first problem, derive the differential form
(3) Specialise the equation for a two-dimensional problem.
(4) Derive the equation in polar coordinates.

5.3.4. Non-dimensional Form of Equations. So far, in this book, we have not talked of the physical units used. How do the equations depend on physical units that we use. Does the solution depend on the fact that we use millimetres instead of metres? We would like to solve the non-dimensional form of these equations. We will demonstrate the process of obtain the non-dimensional form of the equation using the two-dimensional Euler's equation written in Cartesian coordinates.

$$(5.3.37) \qquad \frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0$$

To this end, we define the following reference parameters and relationships. It should be noted that the whole aim of this choice is to retain the form of the equations.

We have a characteristic length L in the problem that we will use to scale lengths and coordinates. For example

$$(5.3.38) \qquad x^* = \frac{x}{L}, \qquad \text{and} \qquad y^* = \frac{y}{L}$$

We employ a reference density ρ_r and a reference pressure p_r to non-dimensionlise the density and the pressure. As a result we get the non-dimensionalisation for the temperature through the equation of state.

(5.3.39)
$$\rho^* = \frac{\rho}{\rho_r}, \qquad \text{and} \qquad p^* = \frac{p}{p_r}, \qquad \text{along with} \qquad p = \rho RT \qquad \text{gives} \qquad T^* = \frac{T}{T_r},$$

where,

$$(5.3.40) \qquad T_r = \frac{p_r}{\rho_r R},$$

and the equation of state reduces to

$$(5.3.41) \qquad p^* = \rho^* T^*$$

Consider the one-dimensional energy equation from gas dynamics. This relation tells us that

$$(5.3.42) \qquad\qquad C_p T_o = C_p T + \frac{V^2}{2}$$

If we divide this equation through by T_r and nondimensionalise speed with a reference speed u_r we get

$$(5.3.43) \qquad\qquad C_p T_o^* = C_P T^* + \frac{V^{*2} u_r^2}{2T_r}$$

Now we see that if we define

$$(5.3.44) \qquad\qquad u_r = \sqrt{RT_r}$$

equation (5.3.43) reduces to

$$(5.3.45) \qquad\qquad \frac{\gamma}{\gamma - 1} T_o^* = \frac{\gamma}{\gamma - 1} T^* + \frac{V^{*2}}{2}$$

Now, consider the first equation, conservation of mass, from equations (5.3.37). This becomes

$$(5.3.46) \qquad\qquad \frac{\rho_r}{\tau} \frac{\partial \rho^*}{\partial t^*} + \frac{\rho_r u_r}{L} \frac{\partial \rho^* u^*}{\partial x^*} + \frac{\rho_r u_r}{L} \frac{\partial \rho^* v^*}{\partial y^*} = 0$$

where τ is some characteristic time scale to be defined here. Dividing through by $\rho_r u_r$ and multiplying through by L, we get

$$(5.3.47) \qquad\qquad \frac{L}{u_r \tau} \frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} + \frac{\partial \rho^* v^*}{\partial y^*} = 0$$

Clearly, if we define the time scale $\tau = L/u_r$ we get back our original equation. I will leave it as an exercise in calculus for the student to show that given the following summary

$$(5.3.48) \qquad\qquad x^* = \frac{x}{L}, \qquad \text{and} \qquad y^* = \frac{y}{L}$$

$$(5.3.49) \qquad\qquad \rho^* = \frac{\rho}{\rho_r}, \qquad \text{and} \qquad p^* = \frac{p}{p_r},$$

$$(5.3.50) \qquad\qquad T_r = \frac{p_r}{\rho_r R}, \qquad \text{and} \qquad u_r = \sqrt{RT_r}$$

equation (5.3.37) reduces to

$$(5.3.51) \qquad\qquad \frac{\partial Q^*}{\partial t^*} + \frac{\partial E^*}{\partial x^*} + \frac{\partial F^*}{\partial y^*} = 0$$

where

$$(5.3.52)$$
$$Q^* = \begin{bmatrix} \rho^* \\ \rho^* u^* \\ \rho^* v^* \\ \rho^* E_t^* \end{bmatrix}, \qquad E^* = \begin{bmatrix} \rho^* u^* \\ \rho^* u^{*2} + p^* \\ \rho^* u^* v^* \\ (\rho^* E_t^* + p^*) u^* \end{bmatrix}, \qquad \text{and} \qquad F^* = \begin{bmatrix} \rho^* v^* \\ \rho^* u^* v^* \\ \rho^* v^{*2} + p^* \\ (\rho^* E_t^* + p^*) v^* \end{bmatrix}$$

Very often for the sake of convenience the "stars" are dropped. One has to remember that though these basic equations have not changed form. Others have changed form. The equation of state becomes $p^* = \rho^* T^*$ and the energy equation changes form. Any other auxiliary equation that you may use has to be nondimensionalised using the same reference quantities.

A careful study will show you that if L, p_r and ρ_r are specified then all the other reference quantities can be derived. In fact, we typically need to fix two reference quantities along with a length scale and the others can be determined. The other point to note is that we typically pick reference quantities based on the problem at hand. A review of dimensional analysis at this point would be helpful.

Assignment 5.4

(1) Non-dimensionalise the Euler's equation in the differential form for three-dimensional flows.
(2) Try to non-dimensionalise the Burgers' equation

$$(5.3.53) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

u does not have the units of speed.