# AS2070: Aerospace Structural Mechanics

## Module 3: Introduction to Fatigue and Failure (V2)

**Instructor: Nidish Narayanaa Balaji**

Department of Aerospace Engineering, IIT Madras

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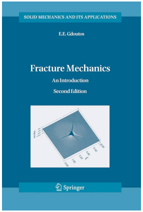
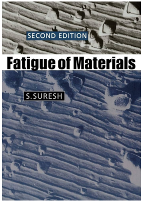
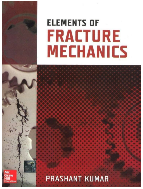
Also see https://www.fracturemechanics.org/



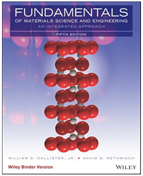

*Chapters 1,4 in Gdoutos (2005)*



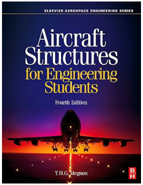

*Chapters 1,7,9 in Suresh (1998)*




*Chapters 1-3 in Kumar (2009)*



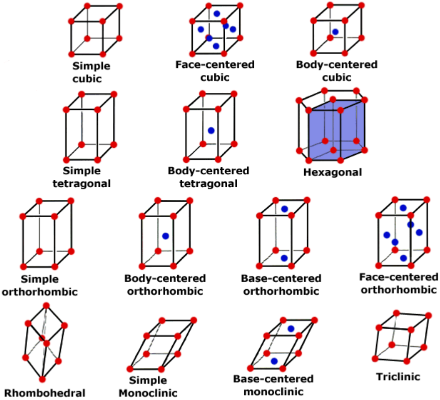

*Chapter 3 in Jr and Rethwisch (2012)*




*Chapter 15 in Megson (2013)*

# 1.1. Structure of Materials

## Introduction




*Types of crystal structures in metals Sparky (2013)*

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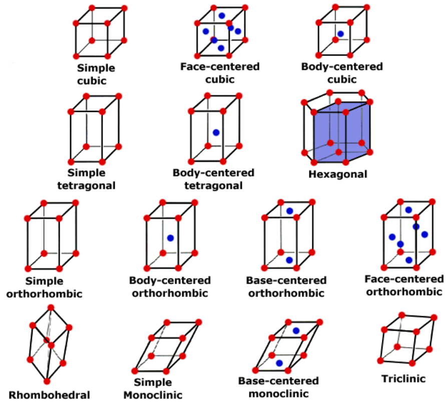


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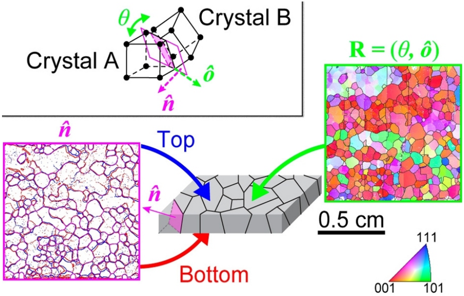


*Crystal and Grain Structures New Technique Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"*

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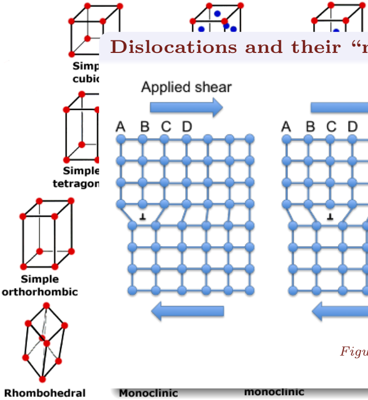



Types of crystal structures in metals Sparky (2013)

### Dislocations and their "motion" under load

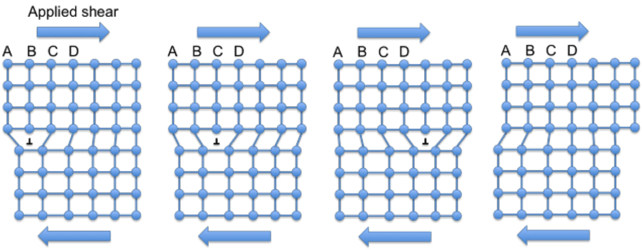
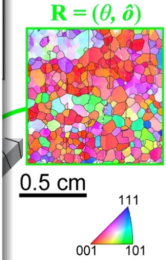



*Figure from Sparky (2013)*




*...que Provides Detailed Views of Metals' Crystal Structure (2016). "Polycrystallinity"*

# 1.2. Understanding the Stress-Strain Curve

Introduction

**The Uniaxial Tensile Test**

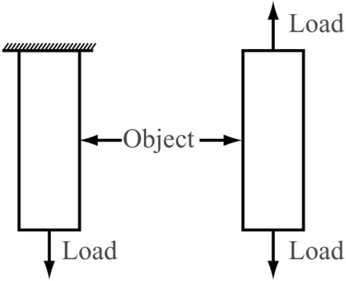



*Figure from Rajendran 2011*

# 1.2. Understanding the Stress-Strain Curve

Introduction

## Terminology

1. Proportionality Limit;
2. Elastic Limit;
3. Yield Point;
4. Ultimate Strength;
5. Fracture Point;
6. Elongation at Failure;

## Ductile Fracture

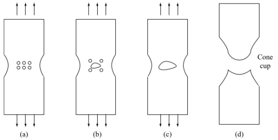



*Figure from Rajendran 2011*

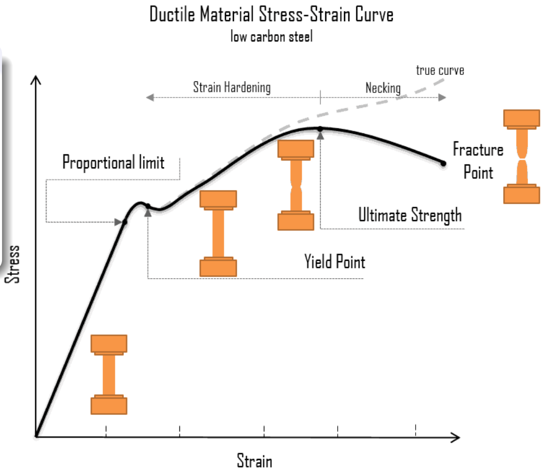



*Figure from Connor 2020*

# 1.3. Failure Mechanisms: Fracture

1. Introduction

**"Griffith Theory" of brittle fracture**

- Theoretical fracture stress $\sim \frac{E}{5} - \frac{E}{30}$ (steel $\sim \frac{E}{1000}$)
- Fracture occurs when $E_{strain} = E_{surface}$
- Crack propagates when $\frac{dE_{strain}}{dL} = \frac{dE_{surface}}{dL}$

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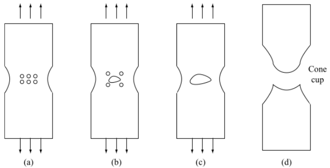


*Ductile Fracture Rajendran 2011*

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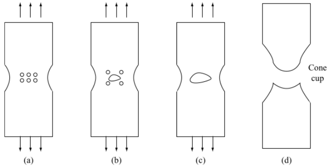



*Ductile Fracture Rajendran 2011*

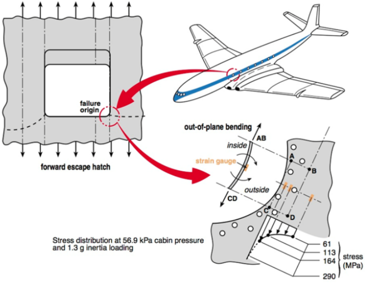
| Sr. No | Brittle Fracture | Ductile Fracture |
|---|---|---|
| 1. | It occurs with no or little plastic deformation. | It occurs with large plastic deformation. |
| 2. | The rate of propagation of the crack is fast. | The rate of propagation of the crack is slow. |
| 3. | It occurs suddenly without any warning. | It occurs slowly. |
| 4. | The fractured surface is flat. | The fractured surface has rough contour and the shape is similar to cup and cone arrangement. |
| 5. | The fractured surface appears shiny. | The fractured surface is dull when viewed with naked eye and the surface has dimpled appearance when viewed with scanning electron microscope. |
| 6. | It occurs where micro crack is larger. | It occurs in localised region where the deformation is larger. |

*Ductile vs Brittle Fracture Rajendran 2011*

# 1.3. Failure Mechanisms: Fatigue

1. Introduction

..over 90% of mechanical failures are caused because of metal fatigue *What Is Metal Fatigue? 2021*...




*The De Havilland Comet The deHavilland Comet Disaster 2019 [lecture]*

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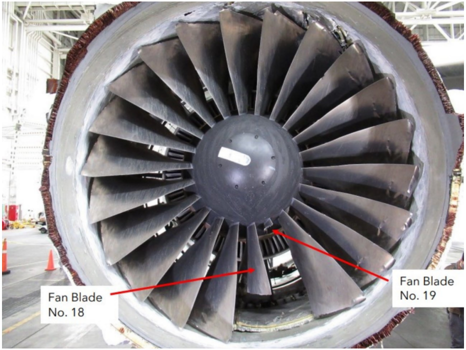
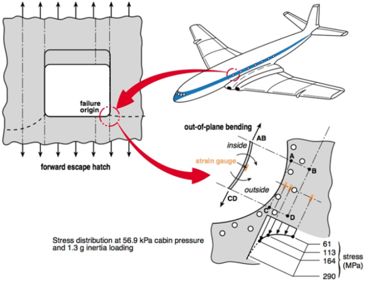



*A more recent example (2021 United Airlines Boeing 777) DCA21FA085.Aspx 2024. [video]*

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*A more recent exampl*
*777) DCA21FA085.Aspx 2024. [video]*

### Fatigue Crack Propagation: Beech Marks

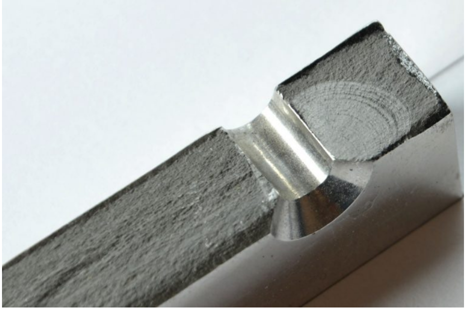
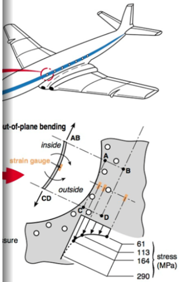


*Figure from Fatigue Physics 2024*

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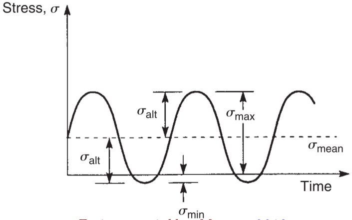
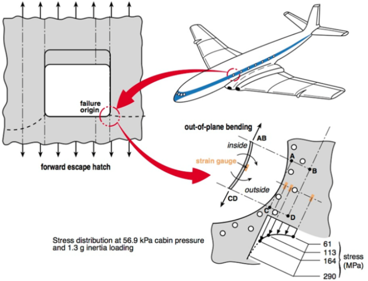



*Fatigue variables Megson 2013*

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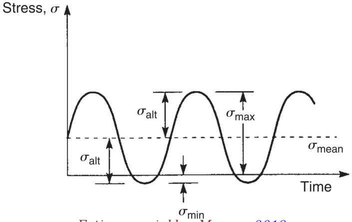


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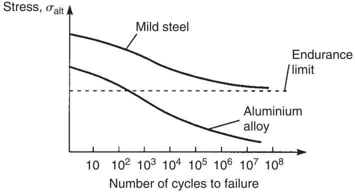
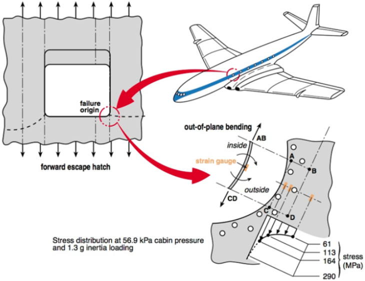



*The S-n Diagram Megson 2013*

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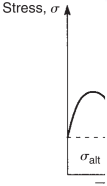


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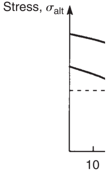


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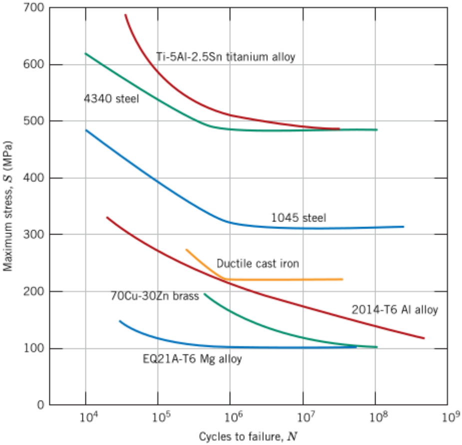
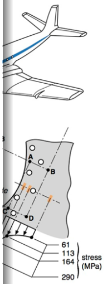
## S-N Curves for Common Metals ( Jr and Rethwisch 2012)

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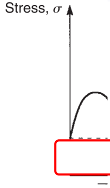


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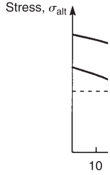
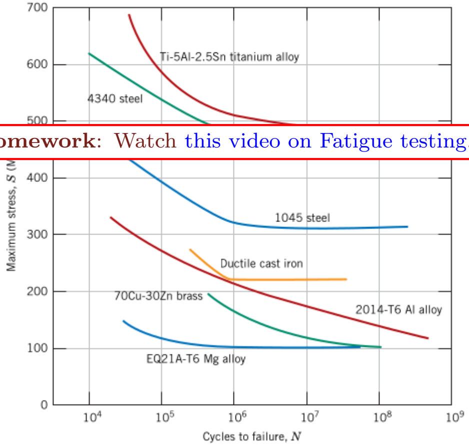
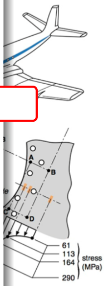



*The*

**S-N Curves for Common Metals** ( Jr and Rethwisch 2012)



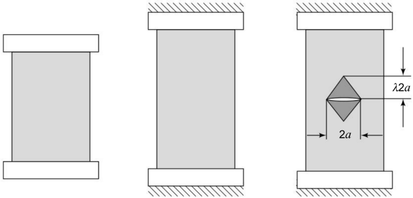

**Homework**: Watch this video on Fatigue testing.




*d Comet Disaster*

# 1.4. Energy Release Rate: Griffith's Analysis

Introduction




*Simplistic picture of the introduction of a crack in a stretched specimen(Figure from sec 2.5 in Kumar 2009)*

- Because of the crack, we assume $\sigma \approx 0$ in the triangles.
- Corresponding energy loss:

$$E_R = V_\Delta \times (\frac{\sigma^2}{2E}) = \frac{2a^2 \lambda t \sigma^2}{E}.$$

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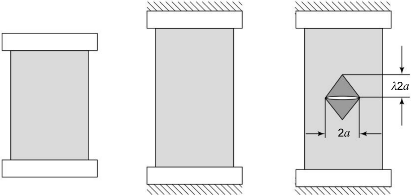


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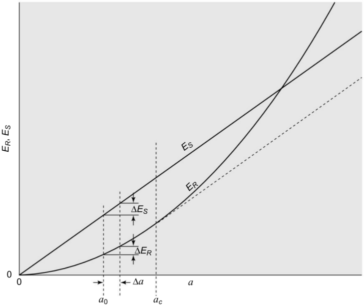
- For thin plates, $\lambda = \frac{\pi}{2}$. So,

$$E_R = \frac{\pi a^2 t\sigma^2}{E}.$$

- The "creation" of a surface takes energy. We write this as,

$$E_S = 2(2at)\gamma = 4at\gamma.$$




*(Figure from Kumar 2009)*

# 1.4. Energy Release Rate: Griffith's Analysis

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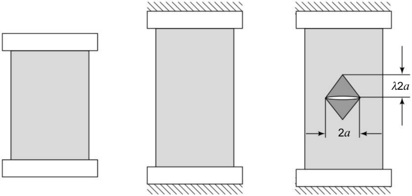


*Simplistic picture of*
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**Food For Thought**

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*

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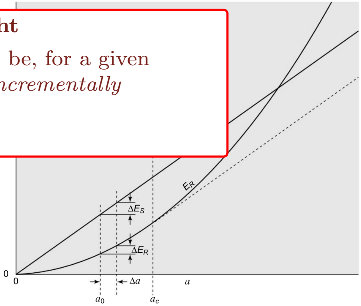


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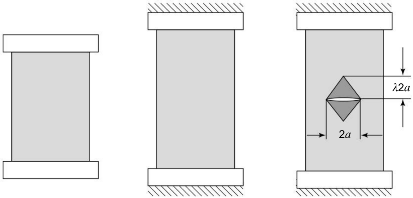


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**Food For Thought**

- What would a "safe size" of crack be, for a given loading condition? *Hint: Think incrementally*
- What type of an experiment would be necessary to confirm this mathematical framework?

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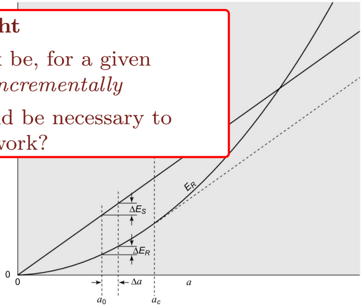



*(Figure from Kumar 2009)*

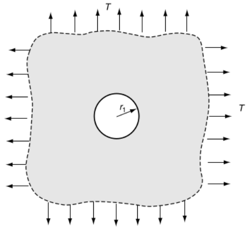
# 1.5. Linear Elastic Fracture Mechanics

Introduction

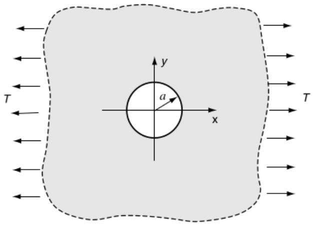
(Ref: Sec. 8.4.2 in Sadd 2009)

Consider the following two cases.
**Question**: Where will the point of peak stress occur? And which will have higher maximum stress?

**Case 1**




**Case 2**



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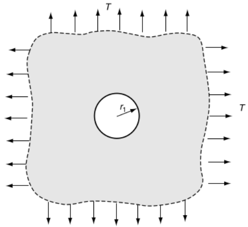


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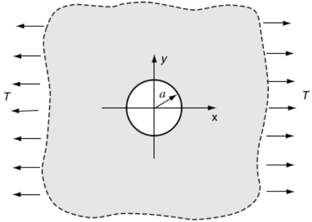


**Analytical Solution**

$\sigma_r = T(1 - \frac{r_1^2}{r^2}), \ \sigma_\theta = T(1 + \frac{r_1^2}{r^2})$

$\implies \boxed{\sigma_{\max} = 2T}$

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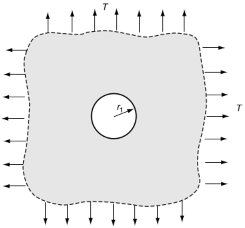
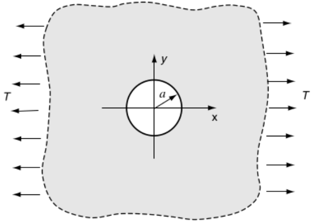



**Case 2**




**Analytical Solution**

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**Analytical Solution**

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \ldots$
$\implies \boxed{\sigma_{\max} = 3T}$

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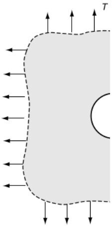
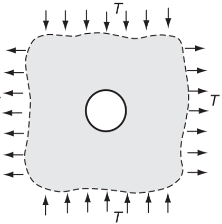
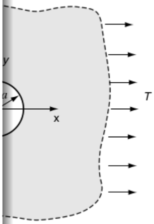



### Case 3




$$\sigma_{\max} = 4T$$



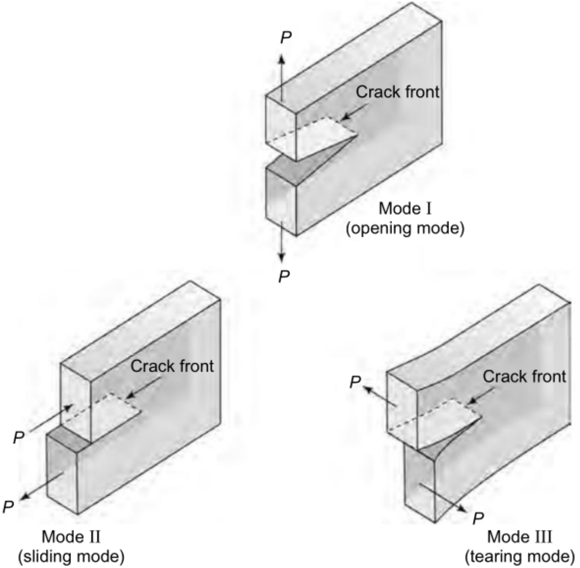

### Analytical Solution

$\sigma_r = T(1 - \frac{r_1^2}{r^2})$, $\sigma_\theta = T(1 + \frac{r_1^2}{r^2})$
$\implies$ $\boxed{\sigma_{\max} = 2T}$

$\sigma_r = T(1 - \frac{r_1^2}{r^2}) + (\cdot)\cos(2\theta)$, $\sigma_\theta = \dots$
$\implies$ $\boxed{\sigma_{\max} = 3T}$

# 1.6. Modes of Fracture

Introduction



# 2. Introduction to Failure

**Necessary Reading**: Secs. 5-3 to 5-9 in Budynas, Nisbett, and Shigley 2015

- Elasticity only deals with the **reversible deformation** behavior. But real materials undergo yielding, necking, and eventually failure under large loads.
- Since there is permanent loss during the yield process (irreversibility), we will have to abandon **non-dissipativity** and modify the stress-strain relationship.

**An Approach for a Failure Theory...**

- **Conduct** controlled experiments in the lab
- **Observe** characteristics of failure
- **Propose generalizations** and test with further experimentation

*Failed mild steel specimens (under uni-axial tension)*

# 2. Introduction to Failure

- A first observation is that materials can be classified as brittle or ductile based on whether they undergo a plastic yield before failure.
- Generally, a material is said to be

  **Ductile** if strain at failure is greater than 5 %,
  **Brittle** if strain at failure is lesser than 5 %.
- For **Ductile Materials** the commonly applicable theories are
  - **Maximum Shear Stress Theory (MSS)**: Failure occurs when the maximum shear stress reaches a threshold. (aka **Tresca theory**)
  - **Distortion Energy Theory (DE)**: Failure occurs when the distortional strain energy reaches a threshold.
  - **Ductile Coulomb-Mohr (DCM) Theory**: The shear stress threshold for failure grows linearly with straight stress. *Proposed to account for tensile and compressive strengths being different*
- For **Brittle Materials** the applicable theories are
  - **Maximum Normal Stress Theory (MNS)**: Failure occurs when the normal stress reaches a threshold.
  - **Brittle Columb-Mohr (BCM) Theory**: Same idea as in DCM, to account for different tensile and compressive strengths.
- We will only bother ourselves with MSS, DE, and MNS here.

# 2.1. Maximum Shear Stress Theory

Introduction to Failure

- For ductile materials, failure in a uniaxial tensile test almost always involves a plane at 45° from the direction of loading.
- From the Mohr circle relationships for the uniaxial case at yielding point ($\sigma_1 = S_y, \sigma_2 = 0$) we have,

$$\sigma_n = \frac{S_y}{2} + \frac{S_y}{2}\cos 2\theta, \quad \tau_s = -\frac{S_y}{2}\sin 2\theta.$$

- **We empirically observe that the $\theta = 45°$ also corresponds to the plane where the shear traction component is maximum!**

**We hypothesize that**

*...yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield.*

*Failed mild steel specimens (under uni-axial tension)*

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Note: Critical shear stress is $S_{sy} = S_y/2 = 0.5S_y$

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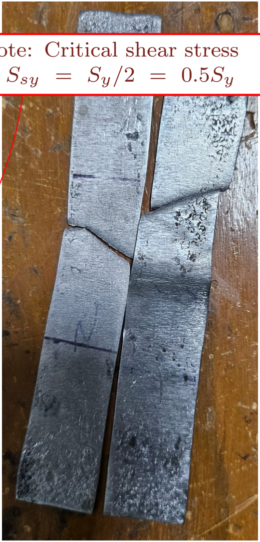

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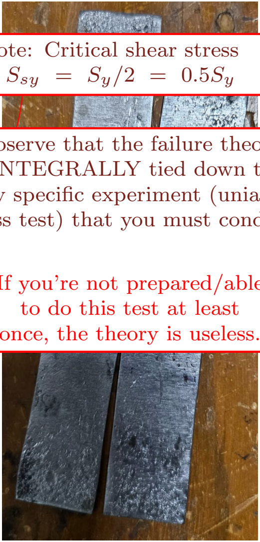
Note: Critical shear stress is $S_{sy} = S_y/2 = 0.5S_y$

Observe that the failure theory is INTEGRALLY tied down to a very specific experiment (uniaxial stress test) that you must conduct.

If you're not prepared/able to do this test at least once, the theory is useless.



*Failed mild steel specimens (under uni-axial tension)*

# 2.1. Maximum Shear Stress Theory: Application to Bi-Axial Loading



- For a general case, all of $\sigma_1$, $\sigma_2$ and $\sigma_3$ are non-zero.
- The maximum shear stress that can be achieved (among all possible planes) is $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$. MSS predicts that failure will happen when $\tau_{\max} \geq 0.5 S_y$, i.e.,

$$\boxed{|\sigma_1 - \sigma_2| \geq S_y}, \boxed{|\sigma_1 - \sigma_3| \geq S_y}, \boxed{|\sigma_2 - \sigma_3| \geq S_y}.$$

  Note that for the 2D case we will take $\sigma_3 = 0$.
- We've got the "non-yield region" in the $\sigma_1, \sigma_2$ space to be:

$$\{(\sigma_1, \sigma_2) \quad | \quad |\sigma_1 - \sigma_2| < S_y, |\sigma_1| < S_y, |\sigma_2| < S_y\}$$

- **So given a general state of stress $\underset{\approx}{\sigma}$, we must first estimate the principal stresses and check if they fall within the above to check if it would fail according to the MSS/Tresca theory.**

# 2.1. Maximum Shear Stress Theory: Application to Bi-Axial Loading

Introduction to Failure

- For a ge
- The max ) is $\tau_{\max} =$ i.e.,

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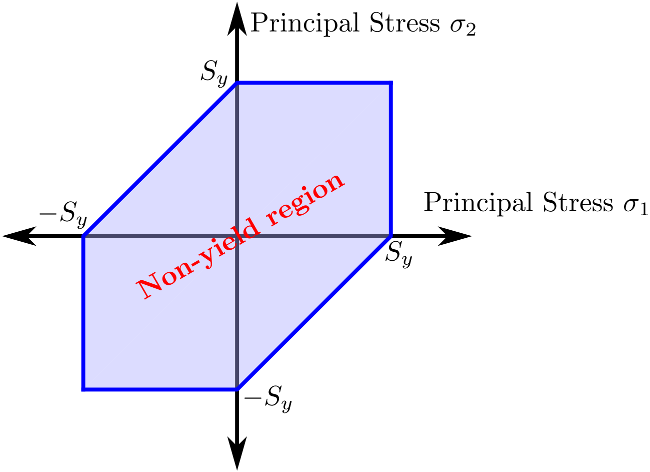
**Graphical Depiction of the Non-Yield Region from MSS**



# 2.2. Distortion Energy Theory

Introduction to Failure

- We **hypothesize here** that shape change energy is what leads to failure, **NOT isotropic volume change**.
- So we "remove" the volume change energy from the overall energy and require that the remaining energy (which we will call as "distortional" since this only represents shape change) does not exceed the energy contained in a uniaxial test context.

**Formally we hypothesize that**

*yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.*

# 2.2. Distortion Energy Theory

Introduction to Failure

- The total energy for a general state of stress in an isotropic material is given by:

$$\mathcal{U} = \frac{1}{2E_Y}\begin{bmatrix}\sigma_1 & \sigma_2 & \sigma_3\end{bmatrix}\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \frac{1}{2E_Y}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)\right)$$

- The work done by the "hydrostatic pressure" is written by replacing each $\sigma_i$ by $\sigma_{av} = \frac{\sigma_1+\sigma_2+\sigma_3}{3}$:

$$\mathcal{U}_v = \frac{3(1-2\nu)}{2E_Y}\sigma_{av}^2 = \frac{1-2\nu}{6E_Y}(\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)).$$

  $\mathcal{U}_v$ is interpreted as energy that only goes into volume change.
- The shape change/distortional energy is the "remaining energy" after taking $\mathcal{U}_v$ out of $\mathcal{U}$:

$$\mathcal{U}_d = \mathcal{U} - \mathcal{U}_v = \frac{1+\nu}{E_Y}\left(\frac{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}{2}\right).$$

- A uniaxial tensile test specimen fails when the state of stress is $(\sigma_1,\sigma_2,\sigma_3) = (S_y, 0, 0)$. The corresponding distortional energy is written as

$$\mathcal{U}_{d,y} = \frac{1+\nu}{E_Y}S_y^2 \implies \boxed{\mathcal{U}_d < \mathcal{U}_{d,y}} \text{ to avoid yielding.}$$

# 2.2. Distortion Energy Theory

Introduction to Failure

**Take a Second to Interpret!**

- Mathematical Summary:

$$\boxed{\sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}{2}} < S_y}.$$

- An alternative interpretation of this theory comes from the fact that the shear stress magnitude on an octahedral plane (plane with normal $\underset{\sim}{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$) is the same as the LHS above.
- Sometimes referred to as the **Maximum Octahedral Stress Theory**, this states that failure happens due to shear on an octahedral plane. It is mathematically identical to DE.
- It is interesting to note that the epistemology of failure is actually not our concern here - we are merely interested in putting our uniaxial tensile testing data to good use in a way that captures the uniaxial tensile test itself as just a sub-case !

$$\mathcal{U}_{d,y} = \frac{1+\nu}{E_Y} S_y^2 \implies \boxed{\mathcal{U}_d < \mathcal{U}_{d,y}} \text{ to avoid yielding.}$$

# 2.2. Distortion Energy Theory

Introduction to Failure

- For the plane stress situation $(\sigma_1 \neq 0, \sigma_2 \neq 0, \sigma_3 = 0)$, the non-yield region is given by

$$\{(\sigma_1, \sigma_2) \quad | \quad \frac{(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2}{2} \leq S_y^2\},$$

  which defines a rotated ellipse in 2D.
- For a state of "pure shear" of magnitude $\tau$, the principal stresses are $\sigma_1 = \tau$, $\sigma_2 = -\tau$. Here, the criterion becomes,

$$3\tau^2 \leq S_y^2 \implies S_{sy} = \frac{S_y}{\sqrt{3}} = 0.577 S_y.$$

# 2.2. Distortion Energy Theory

Introduction to Failure

- For the p … is given by

  which de …

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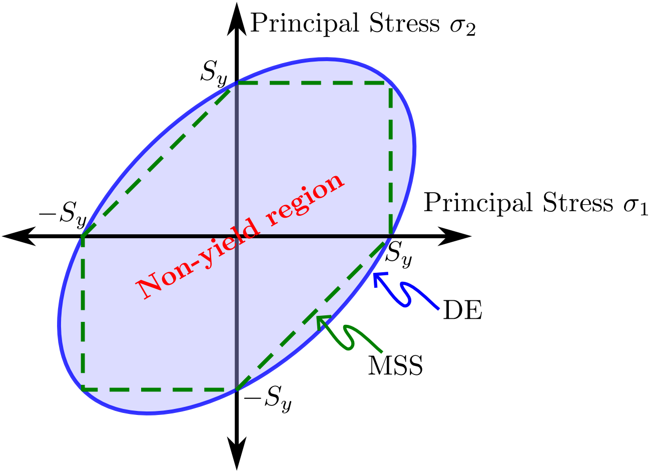
**Graphical Depiction of the Non-Yield Region from DE & MSS**



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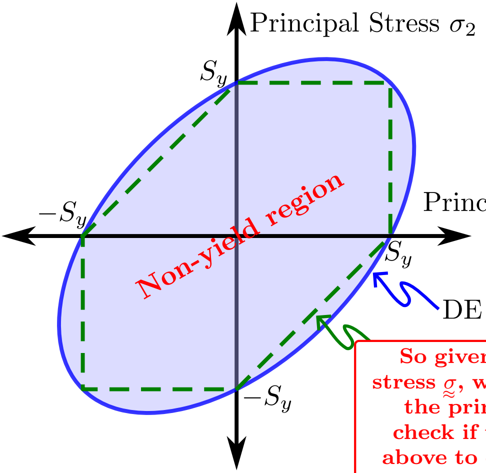


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$$3\tau^2 \leq S_y^2 \implies S_{sy} = \frac{S_y}{\sqrt{3}} = 0.577 S_y.$$

What about **Plane Strain**? Will there be any difference?

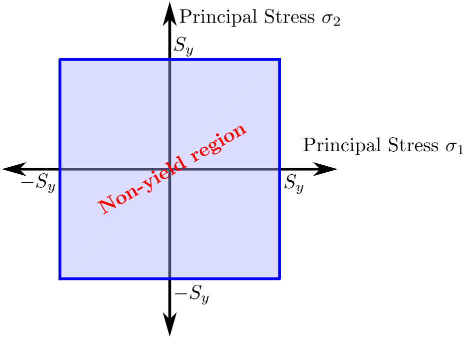
# 2.3. Maximum Normal Stress Theory

Introduction to Failure

**We hypothesize that**

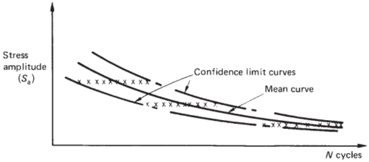
yielding occurs when the **maximum normal stress** in any element equals or exceeds the maximum normal stress in a tension-test specimen of the same material when that specimen begins to yield.

- The hypothesis follows in the same vein as the previous two theories - following experimental experience with brittle material fracture which happens, on tensile test specimens, on a plane perpendicular to the direction of loading.



# 3. Introduction to Fatigue

- It is sometimes the case that components in machinery fail even though no part of them has ever experienced stresses that violated any of the above failure theory envelopes!
- This is known as **Fatigue**, and pertains to failure under repetitive stress cycles.
- Remind yourself that we are **NOT** very much concerned with the epistemology (they "why") of this phenomenon: We have observed this, and we want to use this knowledge in design. This will be our approach.
- Plotting the **stress-amplitude-to-failure** against the **number-of-cycles-to-failure** leads to the famous "S-n Curve" of fatigue.



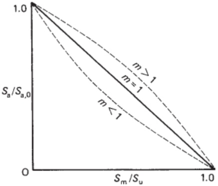

*(Figure 15.1 from Megson 2013)*

# 3. Introduction to Fatigue

### Concepts

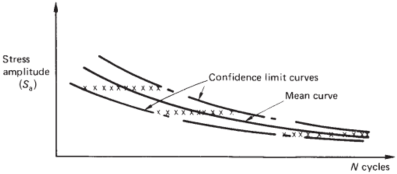
- Safe Life: RUL
- Fail-Safe: Redundancy

### Tensile Stresses: The Goodman Diagram



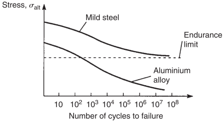

*(Figure 15.2 from Megson 2013)*

$$\frac{S_a}{S_{a,0}} = 1 - \left(\frac{S_m}{S_u}\right)$$




*(Figure 15.1 from Megson 2013)*

### The S-N Curve




*(Figure from Megson 2013)*

$$\sigma_{alt} = \sigma_\infty \left(1 + \frac{C}{\sqrt{N}}\right), \quad N \propto \frac{1}{\sigma_{mean}}.$$

# 3.1. The deHavilland Comet

Introduction to Fatigue

*No aircraft has contributed more to safety in the jet age than the Comet. The lessons it taught the world of aeronautics live in every jet airliner flying today. – D.D. Dempster, 1959, in The Tale of the Comet*

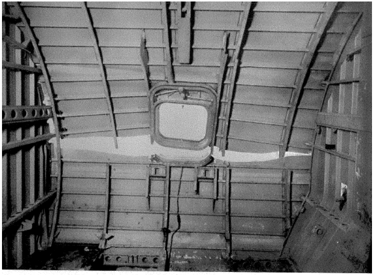



FIG. 7. VIEW FROM INSIDE OF FAILURE AT THE FORWARD ESCAPE HATCH ON THE PORT SIDE—COMET G-ALYU

*(Figures from "De Havilland Comet" 2025)*

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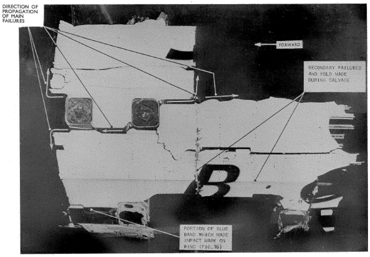
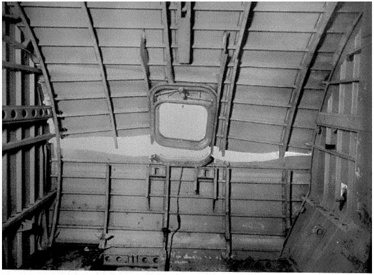



FIG. 12. PHOTOGRAPH OF WRECKAGE AROUND ADF AERIAL WINDOWS—G-ALYP.



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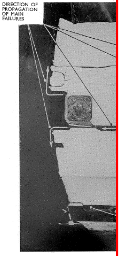
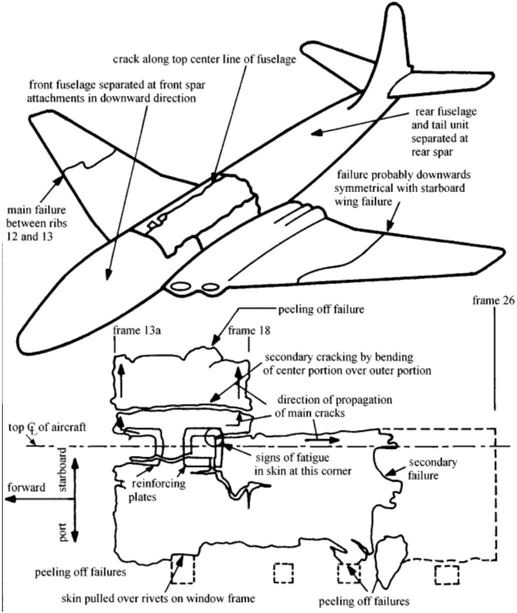



FIG. 12. PHOTO

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t” 2025)

# 3.2. Miner's Rule

Introduction to Fatigue

- Suppose at an operation level of $\sigma_m, \sigma_a$, the fatigue life is $N$ and the structure undergoes $n$ cycles, Miner's rule posits that $\frac{n}{N}$ is the fraction of life that has been consumed.
- Suppose a structure undergoes multiple stress levels in its loading history, the total fraction of fatigue life that has been consumed is written as

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots.$$

- The structure is expected to fail when this sum becomes 1.0..

# 4.1. Griffith's Analysis and Energy Release Rate

Linear Elastic Fracture Mechanics

- The total energy of a loaded elastic body is written as

$$\Pi = \underbrace{U}_{\text{elastic}} - \underbrace{W}_{\text{external}}.$$

- **Griffith's principle**: The **energy lost due to the creation** of a cracked surface must be equal to the **energy required for the creation** of the cracked surface.
- Surface energy is usually expressed as $E_S = \mathcal{A}\gamma$.
- This is a general principle agnostic of the exact structure under consideration.

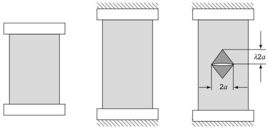
$$\boxed{G = -\frac{d\Pi}{d\mathcal{A}} = 2\gamma}.$$

(note: $2\mathcal{A}$ is the effective total "new" surface area that has been created)

# 4.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

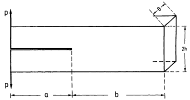
## Crack in Stretched Specimen




*(Figure from sec 2.5 in Kumar 2009)*

- Crack: $\mathcal{A} = 2at$, $\partial_{\mathcal{A}} = \frac{1}{2t}\partial_a$
- $\Pi = U = \frac{\sigma^2 t}{2E'}(\mathcal{A}_{tot} - 4\lambda a^2)$.
- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} = 2\gamma$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}\frac{d\Pi}{da} = \frac{\lambda a}{2E'}\sigma^2$.
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{\frac{2E'\gamma}{\pi a}}$.

## Double Cantilever Beam (DCB)



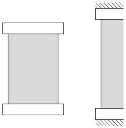

*(Figure 4.14 in Gdoutos 2005)*

- $u = CP = \frac{2a^3}{3EI}P$, $C = \frac{2a^3}{3EI}$.
- $U = \frac{Pu}{2} = \frac{CP^2}{2} = \frac{P^2}{3EI}a^3$,
  $W = Pu = CP^2 = \frac{2P^2}{3EI}a^3$,
  $\Pi = -\frac{P^2}{2}C = -\frac{P^2}{3EI}a^3$.
- $\mathcal{A} = aB$, $\partial_{\mathcal{A}} = \frac{1}{B}\partial_a$.
- $G = -\frac{d\Pi}{d\mathcal{A}} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2 a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

# 4.1. Griffith's Analysis and Energy Release Rate: Examples

Linear Elastic Fracture Mechanics

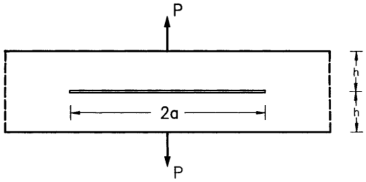
## Crack in Stre



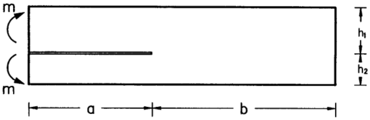
(Figure from sec

- Crack: $\mathcal{A} = 2at$, $\delta$
- $\Pi = U = \frac{\sigma^2 t}{2E'}(\mathcal{A}_{to}$
- $E_S = 2\mathcal{A}\gamma$, $\frac{dE_S}{d\mathcal{A}} =$
- $G = -\frac{d\Pi}{d\mathcal{A}} = -\frac{1}{2t}$
- $\sigma_{cr} = \sqrt{\frac{E'\gamma}{\lambda a}} = \sqrt{}$

## Additional Cases to Consider

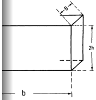



(Figure 4.23 from Gdoutos (2005))




(Figure 4.20 from Gdoutos (2005))

## er Beam (DCB)




Gdoutos 2005)

$= \frac{2a^3}{3EI}$.

$\frac{P^2}{3EI}a^3$,

$\frac{2P^2}{3EI}a^3$,

$a^3$.

.

$= \frac{P^2a^2}{EIB} = \frac{12P^2a^2}{EB^2h^3}$

# 4.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

- In 2D, the governing equations of elasticity (let us assume no body loads for simplicity) are written as,

$$\sigma_{x,x} + \tau_{xy,y} = 0, \quad \tau_{xy,x} + \sigma_{y,y} = 0.$$

- If we seek to obtain **solutions expressed directly in the stresses**, 2 equations won't cut it (we have 3 unique stresses $\sigma_x, \sigma_y, \tau_{xy}$). So we invoke strain compatibility, which is written as

$$\boxed{\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}}$$

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

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$$\boxed{\varepsilon_{x,yy} + \varepsilon_{y,xx} = \gamma_{xy,xy}}$$

**Recall**: These are conditions that the strains must satisfy in order for them to have been generated by a continuously differentiable displacement field.

- This can be expressed in terms of the stresses if we invoke the **stress-strain constitutive relationships**.

# 4.2. A Primer on 2D Elasticity

Linear Elastic Fracture Mechanics

**Plane Stress**

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

**Compatibility**

$$\sigma_{x,yy} + \sigma_{y,xx} - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2(1+\nu)\tau_{xy,xy}.$$

**Plane Strain**

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

**Compatibility**

$$(1-\nu)(\sigma_{x,yy} + \sigma_{y,xx}) - \nu(\sigma_{x,xx} + \sigma_{y,yy}) = 2\tau_{xy,xy}.$$

- Making the substitution $\sigma_x = \phi_{,yy}$, $\sigma_y = \phi_{,xx}$, $\tau_{xy} = -\phi_{,xy}$, it is trivial to see that the equilibrium equations are satisfied automatically.
- In both the above cases, the compatibility equation comes out to be:

$$\phi_{,xxxx} + \phi_{,yyyy} + 2\phi_{,xxyy} = (\partial_{xx} + \partial_{yy})^2\phi = \nabla^4\phi = 0.$$

- Since the Laplacian when set to zero ($\nabla^2\phi = 0$) is referred to as the **harmonic equation** (recall complex analyticity), $\nabla^4\phi = 0$ is referred to as the **bi-harmonic equation**. $\phi$ is the **Airy Stress Function**.

# 4.3. Classical Solutions

Linear Elastic Fracture Mechanics

- Restricting ourselves to 2D problems, the governing equations may be written using the Airy's stress formulation as the biharmonic equation

$$\nabla^4 \phi = 0$$

- Let us look at this with cylindrical coordinates ($x = r\cos\theta$, $y = r\sin\theta$).

$$\underline{\nabla}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,r}\\ \frac{\phi_{,\theta}}{r}\end{bmatrix}, \quad \underline{\underline{\nabla u}} = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}u_{r,r} & \frac{u_{r,\theta}-u_\theta}{r}\\ u_{\theta,r} & \frac{u_{\theta,\theta}+u_r}{r}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}$$

$$\underline{\underline{\nabla^2}}\phi = \begin{bmatrix}\underline{e}_r & \underline{e}_\theta\end{bmatrix}\begin{bmatrix}\phi_{,rr} & \partial_r(\frac{\phi_{,\theta}}{r})\\ \partial_r(\frac{\phi_{,\theta}}{r}) & \frac{\phi_{,r}}{r}+\frac{\phi_{,\theta\theta}}{r^2}\end{bmatrix}\begin{bmatrix}\underline{e}_r\\ \underline{e}_\theta\end{bmatrix}.$$

- The stresses are expressed (to satisfy equilibrium) as

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r(\frac{\phi_{,\theta}}{r}).$$

# 4.3. Classical Solutions

Linear Elastic Fracture Mechanics

- R … using the A…

**General form of the Airy's Stress Function (Michell Solution, see Barber 2022, Ch. 8-9)**

$$\begin{aligned}\phi =& a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\ & (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\ & (a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r) \cos\theta \\ & (b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r) \sin\theta \\ & \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\ & \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + b_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta.\end{aligned}$$

- L…

- T…

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r^2}, \quad \sigma_{\theta\theta} = \phi_{,rr}, \quad \tau_{r\theta} = -\partial_r\left(\frac{\phi_{,\theta}}{r}\right).$$

# 4.3.1. The Michell Solution: Tabled Expressions

Classical Solutions

## Stress Components

| $\phi$ | $\sigma_{rr}$ | $\sigma_{r\theta}$ | $\sigma_{\theta\theta}$ |
|---|---|---|---|
| $r^2$ | $2$ | $0$ | $2$ |
| $r^2\ln(r)$ | $2\ln(r)+1$ | $0$ | $2\ln(r)+3$ |
| $\ln(r)$ | $1/r^2$ | $0$ | $-1/r^2$ |
| $\theta$ | $0$ | $1/r^2$ | $0$ |
| $r^3\cos\theta$ | $2r\cos\theta$ | $2r\sin\theta$ | $6r\cos\theta$ |
| $r\theta\sin\theta$ | $2\cos\theta/r$ | $0$ | $0$ |
| $r\ln(r)\cos\theta$ | $\cos\theta/r$ | $\sin\theta/r$ | $\cos\theta/r$ |
| $\cos\theta/r$ | $-2\cos\theta/r^3$ | $-2\sin\theta/r^3$ | $2\cos\theta/r^3$ |
| $r^3\sin\theta$ | $2r\sin\theta$ | $-2r\cos\theta$ | $6r\sin\theta$ |
| $r\theta\cos\theta$ | $-2\sin\theta/r$ | $0$ | $0$ |
| $r\ln(r)\sin\theta$ | $\sin\theta/r$ | $-\cos\theta/r$ | $\sin\theta/r$ |
| $\sin\theta/r$ | $-2\sin\theta/r^3$ | $2\cos\theta/r^3$ | $2\sin\theta/r^3$ |
| $r^{n+2}\cos n\theta$ | $-(n+1)(n-2)r^n\cos n\theta$ | $n(n+1)r^n\sin n\theta$ | $(n+1)(n+2)r^n\cos n\theta$ |
| $r^n\cos n\theta$ | $-n(n-1)r^{n-2}\cos n\theta$ | $n(n-1)r^{n-2}\sin n\theta$ | $n(n-1)r^{n-2}\cos n\theta$ |
| $r^{n+2}\sin n\theta$ | $-(n+1)(n-2)r^n\sin n\theta$ | $-n(n+1)r^n\cos n\theta$ | $(n+1)(n+2)r^n\sin n\theta$ |
| $r^n\sin n\theta$ | $-n(n-1)r^{n-2}\sin n\theta$ | $-n(n-1)r^{n-2}\cos n\theta$ | $n(n-1)r^{n-2}\sin n\theta$ |

*(Table 8.1 from Barber 2022)*

We set rigid body motion components to zero for the displacements

## Displacement Components

| $\phi$ | $2\mu u_r$ | $2\mu u_\theta$ |
|---|---|---|
| $r^2$ | $(\kappa-1)r$ | $0$ |
| $r^2\ln(r)$ | $(\kappa-1)r\ln(r)-r$ | $(\kappa+1)r\theta$ |
| $\ln(r)$ | $-1/r$ | $0$ |
| $\theta$ | $0$ | $-1/r$ |
| $r^3\cos\theta$ | $(\kappa-2)r^2\cos\theta$ | $(\kappa+2)r^2\sin\theta$ |
| $r\theta\sin\theta$ | $\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta+(\kappa+1)\ln(r)\cos\theta\}$ | $\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta-(\kappa+1)\ln(r)\sin\theta\}$ |
| $r\ln(r)\cos\theta$ | $\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta+(\kappa-1)\ln(r)\cos\theta\}$ | $\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta-(\kappa-1)\ln(r)\sin\theta\}$ |
| $\cos\theta/r$ | $\cos\theta/r^2$ | $\sin\theta/r^2$ |
| $r^3\sin\theta$ | $(\kappa-2)r^2\sin\theta$ | $-(\kappa+2)r^2\cos\theta$ |
| $r\theta\cos\theta$ | $\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta-(\kappa+1)\ln(r)\sin\theta\}$ | $\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta-(\kappa+1)\ln(r)\cos\theta\}$ |
| $r\ln(r)\sin\theta$ | $\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta+(\kappa-1)\ln(r)\sin\theta\}$ | $\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta+(\kappa-1)\ln(r)\cos\theta\}$ |
| $\sin\theta/r$ | $\sin\theta/r^2$ | $-\cos\theta/r^2$ |
| $r^{n+2}\cos n\theta$ | $(\kappa-n-1)r^{n+1}\cos n\theta$ | $(\kappa+n+1)r^{n+1}\sin n\theta$ |
| $r^n\cos n\theta$ | $-nr^{n-1}\cos n\theta$ | $nr^{n-1}\sin n\theta$ |
| $r^{n+2}\sin n\theta$ | $(\kappa-n-1)r^{n+1}\sin n\theta$ | $-(\kappa+n+1)r^{n+1}\cos n\theta$ |
| $r^n\sin n\theta$ | $-nr^{n-1}\sin n\theta$ | $-nr^{n-1}\cos n\theta$ |

*(Table 9.1 from Barber 2022)*

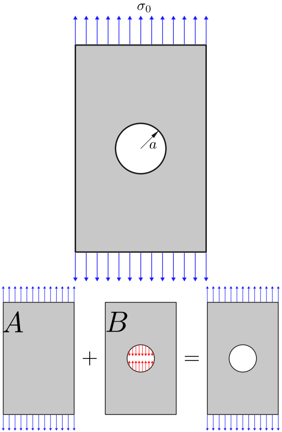
Plane Stress $\kappa = \frac{3-\nu}{1+\nu}$

Plane Strain $\kappa = 3-4\nu$

# 4.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

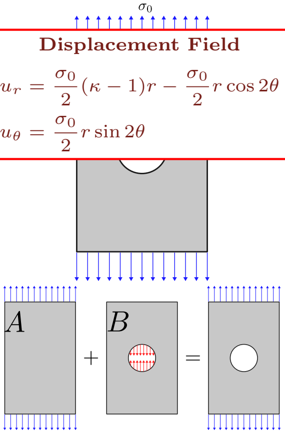
- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.



# 4.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.




**Displacement Field**

$$u_r = \frac{\sigma_0}{2}(\kappa - 1)r - \frac{\sigma_0}{2} r\cos 2\theta$$
$$u_\theta = \frac{\sigma_0}{2} r \sin 2\theta$$

**Problem A**

The 2D stress field (cartesian) is

$$\underline{\underline{\sigma}}_{cart} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix}.$$

- Transforming to cylindrical coordinates,

$$\underline{\underline{\sigma}}_{cyl} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$$
$$= \sigma_0 \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

The components can be written as

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0 \frac{\sin 2\theta}{2},$$
$$\sigma_{\theta\theta} = \sigma_0\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right).$$

# 4.3.Plate with a Hole. Plate With a Hole Under Tension

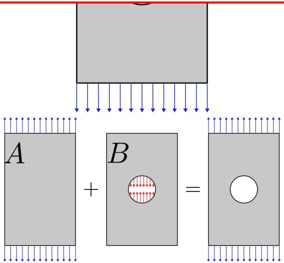
Linear Elastic Fracture Mechanics

- Let us now try to use the above table for obtaining the stress distribution around a hole in a tension field.

**Displacement Field**

$$u_r = \frac{\sigma_0 a^2}{2r}\left(1 - (\kappa + 1 - (\frac{a}{r})^2)\cos 2\theta\right)$$

$$u_\theta = \frac{\sigma_0 a^2}{2r}\left(\kappa - 1 + (\frac{a}{r})^2\right)\sin 2\theta.$$




**Problem B**

At $r = a$ we want

$$\sigma_{rr} = \sigma_0\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right), \quad \sigma_{r\theta} = \sigma_0\frac{\sin 2\theta}{2}.$$

(no hoop component specified)

As $r \to \infty$, we want $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta} \to 0$ to match the far-field.

- Based on inspection (shown in class), we find the following Airy stress function to be a good starting point: $\phi = A\log r + B\theta + C\cos 2\theta + D\frac{\cos 2\theta}{r^2}$.
- Solving for $A, B, C, D$ based on the B.C.s we get,

$$\sigma_{rr} = -\frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + 2\sigma_0\left(\frac{a}{r}\right)^2\left(1 - \frac{3}{4}\left(\frac{a}{r}\right)^2\right)\cos 2\theta,$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(\frac{a}{r}\right)^2 + \frac{3\sigma_0}{4}\left(\frac{a}{r}\right)^4\cos 2\theta,$$

$$\sigma_{r\theta} = \sigma_0\left(\frac{a}{r}\right)^2\left(1 - \frac{3}{2}\left(\frac{a}{r}\right)^2\right)\sin 2\theta$$

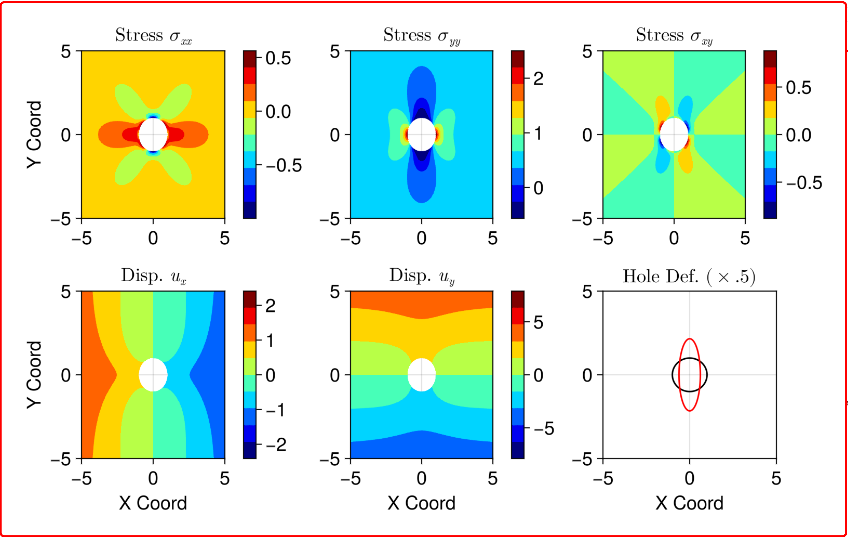
# 4.3.Plate with a Hole. Plate With a Hole Under Tension

Linear Elastic Fracture Mechanics

- L... le
  i...

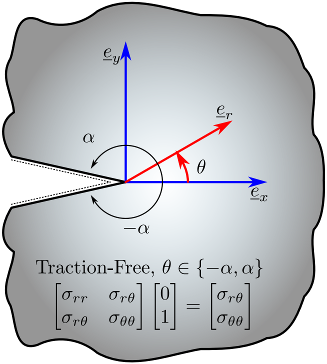
$u_r =$

$u_\theta =$



# 4.3.3. Notch Crack

Classical Solutions




- We seek an analytical solution for this problem setting **very close to the crack**.
- While we may intuitively expect stress to be singular at the crack tip, the strain energy has to be finite.
- Suppose $\sigma \sim \mathcal{O}(r^\lambda)$, $\varepsilon \sim \mathcal{O}(r^\lambda)$ necessarily.
  - So $\mathcal{U} = \int\int \frac{\sigma\varepsilon}{2} r dr d\theta \sim \mathcal{O}(r^{2\lambda+1})$.
  - For this to be finite, $2\lambda + 1 \geq 0 \implies \lambda \geq -\frac{1}{2}$.
- We list out **the only Airy stress functions** that can show this in the following table (refer sl. 27).

| $\phi$ | $\sigma_{rr}$ | $\sigma_{r\theta}$ | $\sigma_{\theta\theta}$ |
|---|---|---|---|
| $r^{n+2}\cos n\theta$ | $(..)r^n \cos n\theta$ | $(..)r^n \sin n\theta$ | $(..)r^n \cos n\theta$ |
| $r^n \cos n\theta$ | $(..)r^{n-2} \cos n\theta$ | $(..)r^{n-2} \sin n\theta$ | $(..)r^{n-2} \cos n\theta$ |
| $r^{n+2}\sin n\theta$ | $(..)r^n \sin n\theta$ | $(..)r^n \cos n\theta$ | $(..)r^n \sin n\theta$ |
| $r^n \sin n\theta$ | $(..)r^{n-2} \sin n\theta$ | $(..)r^{n-2} \cos n\theta$ | $(..)r^{n-2} \sin n\theta$ |

# 4.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

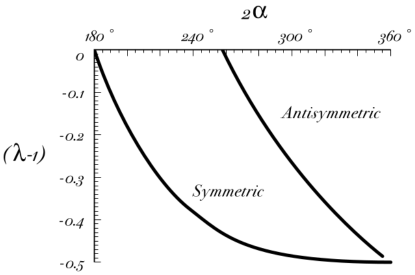
$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

# 4.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$




*(Figure 11.7 from Barber 2022)*

# 4.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

| $\lambda$ | Eigenfunction |
|---|---|
| $\frac{1}{2}$ | $A_2 = \frac{A_1}{3}$, $B_2 = -B_1$ |
| 1 | $A_2 = -A_1$, $B_2 = 0$ $(B_1 = 0)$ |
| $\frac{3}{2}$ | $A_2 = -\frac{A_1}{5}$, $B_2 = -B_1$ |
| $\vdots$ | |

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

# 4.3.3. Singularity Close to Notch Crack

Classical Solutions

- For the Notch crack problem, we posit the Airy stress function

$$\boxed{\phi = r^{\lambda+1}\left(A_1\cos((\lambda-1)\theta) + A_2\cos((\lambda+1)\theta) + B_1\sin((\lambda-1)\theta) + B_2\sin((\lambda+1)\theta)\right)}.$$

- Applying the boundary conditions (along with $\alpha = \pi$), we get a nonlinear eigenvalue problem that has the following solutions:

| $\lambda$ | Eigenfunction |
|---|---|

**Displacement Field**

$$2\mu u_r = K_I\sqrt{\frac{r}{2\pi}}\left((\kappa - \frac{1}{2})\cos\frac{\theta}{2} - \frac{1}{2}\cos\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa - \frac{1}{2})\sin\frac{\theta}{2} - \frac{3}{2}\sin\frac{3\theta}{2}\right)$$

$$2\mu u_\theta = K_I\sqrt{\frac{r}{2\pi}}\left(-(\kappa + \frac{1}{2})\sin\frac{\theta}{2} + \frac{1}{2}\sin\frac{3\theta}{2}\right) - K_{II}\sqrt{\frac{r}{2\pi}}\left((\kappa + \frac{1}{2})\cos\frac{\theta}{2} - \frac{3}{2}\cos\frac{3\theta}{2}\right)$$

- $\lambda = \frac{1}{2}$ corresponds to the near-field singular stress field, given by

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

# 4.3.3. Singularity Close to Notch Crack

Classical Solutions

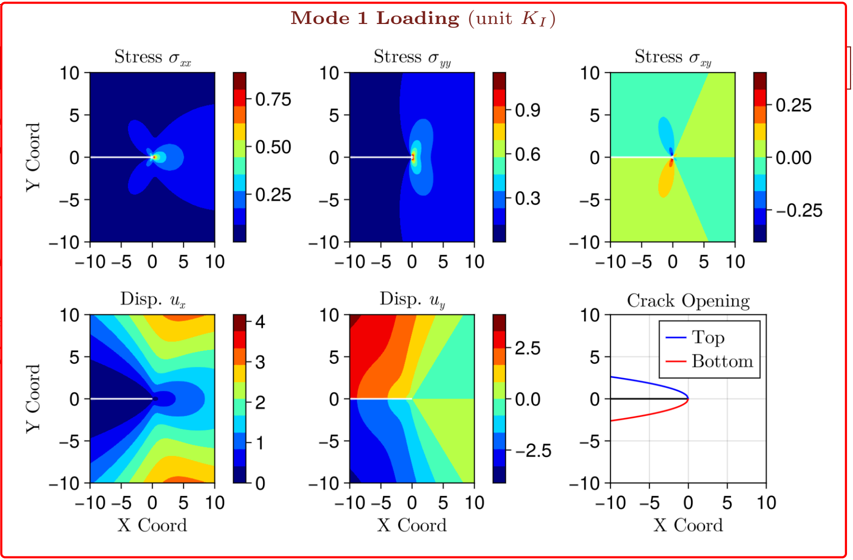
- F

- A
p

$2\mu u$

$2\mu u_\theta$

- $\lambda$



# 4.3.3. Singularity Close to Notch Crack

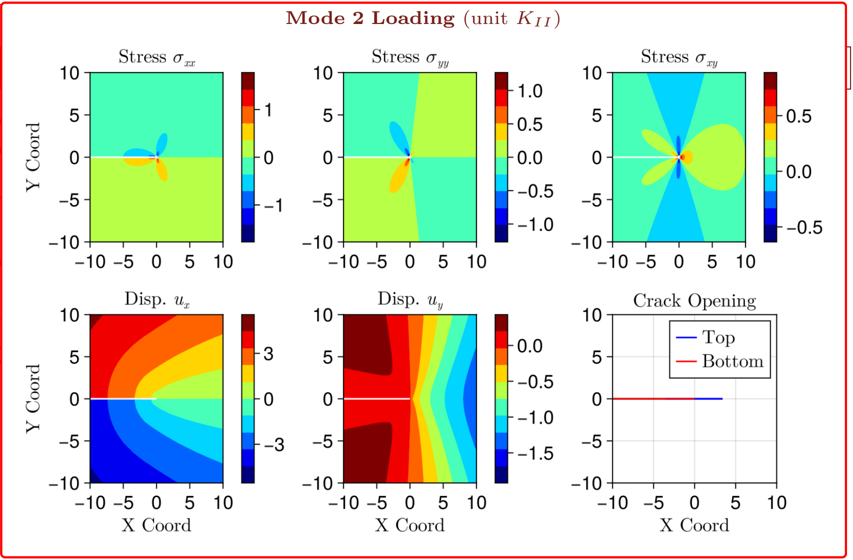
Classical Solutions

- F
- A
  p

$2\mu u$

$2\mu u_\theta$

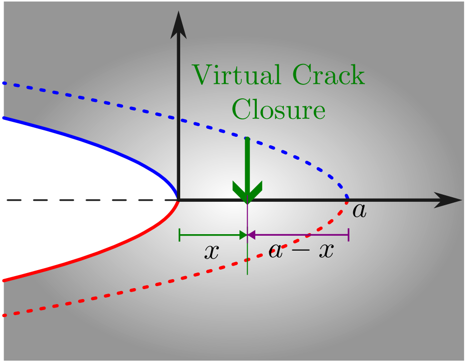
- $\lambda$



# 4.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to “close” a crack.



# 4.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta = 0,\quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi x}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta = \pi,\quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

# 4.3.3. Energy Release Rate

Classical Solutions

- Let us think of how much energy will be necessary to "close" a crack.
- We observe that (all quantities in cylindrical):

$$@\,\theta=0,\quad \underline{\underline{\sigma}} = \frac{K_I}{\sqrt{2\pi r}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi x}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa-1\\ 0\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ \kappa-1\end{bmatrix}$$

$$@\,\theta=\pi,\quad \underline{\underline{\sigma}} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix},\quad 2\mu\underline{u} = K_I\sqrt{\frac{r}{2\pi}}\begin{bmatrix}0\\ -(\kappa+1)\end{bmatrix} - K_{II}\sqrt{\frac{r}{2\pi}}\begin{bmatrix}\kappa+1\\ 0\end{bmatrix}.$$

- For virtual crack closure, the work done can be written as,

$$W(a) = 2\int_0^a \frac{1}{2}\left(\sigma_{\theta\theta}\Big|_{\theta=0}(-u_\theta)\Big|_{\theta=\pi} + \sigma_{r\theta}\big|_{\theta=0}(-u_r)\big|_{\theta=\pi}\right)dx$$

$$= \int_0^a \frac{K_I}{\sqrt{2\pi x}}K_I\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu} + \frac{K_{II}}{\sqrt{2\pi r}}K_{II}\sqrt{\frac{a-x}{2\pi}}\frac{\kappa+1}{2\mu}dx$$

$$= \frac{K_I^2+K_{II}^2}{2\pi}\frac{\kappa+1}{2\mu}\underbrace{\int_0^a\sqrt{\frac{a-x}{x}}dx}_{\frac{a\pi}{2}} = \frac{K_I^2+K_{II}^2}{8\mu^2}(\kappa+1)^2 a = \begin{cases}\frac{K_I^2+K_{II}^2}{E}a & \text{Plane-}\sigma\\ \frac{K_I^2+K_{II}^2}{E}(1-\nu^2)a & \text{Plane-}\varepsilon\end{cases}.$$

- The Griffith Energy Release Rate is the derivative $\lim_{a\to 0}\frac{1}{B}\frac{dW}{da}$, which evaluates as

$$G = \frac{1}{B}\begin{cases}\frac{K_I^2}{E}+\frac{K_{II}^2}{E} & \text{Plane Stress}\\ \frac{K_I^2}{E}(1-\nu^2)+\frac{K_{II}^2}{E}(1-\nu^2) & \text{Plane Strain}\end{cases}.$$

# 4.3.3. Stress Intensity Factor

Classical Solutions

- A crack is said to propagate when $G$ exceeds $G_{cr}$.
- Therefore, under "pure" mode 1 loading, the *Critical Stress Intensity Factor* ($K_{I,cr}$) is

$$K_{I,cr} = \begin{cases} \sqrt{BG_{cr}E} & \text{Plane Stress} \\ \sqrt{\frac{BG_{cr}E}{1-\nu^2}} & \text{Plane Strain} \end{cases}.$$

- **But how do we relate $K_I, K_{II}$ with far-field applied stresses?**

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- **But how do we relate $K_I, K_{II}$ with far-field applied stresses?** The answer is very closely tied in to the exact geometry, loading conditions, etc.

# 4.3.3. Griffith-Inglis Crack Revisited

Classical Solutions

- For the flat crack of length $2a$ (aka the Griffith-Inglis crack), the SIF is related to tensile stresses by

$$K_I = \sigma_0 \sqrt{\pi a}.$$

- Note that this is why we chose $\lambda = \frac{\pi}{2}$ in sl. 7. If we left it in, we'll have to satisfy (plane stress considered here):

$$\frac{4\lambda a}{E}\sigma_0^2 = \frac{2K_I^2}{E} = \frac{2\pi a}{E}\sigma_0^2.$$

# 4.4. Crack Propagation and the Paris Law

Linear Elastic Fracture Mechanics

- **Paris Law**: $\frac{da}{dN} = C(\Delta K)^m$.
- Usually $a_f$ is specified and we are interested in finding how many cycles until a crack of size $a_i$ grows to $a_f$. This is the "life" of the material.

*Values for common engineering materials, from Kumar 2009*

| Material | $C$ | $m$ |
|---:|:---:|:---:|
| Ferrite-Pearlite (S) | $6.8\times10^{-12}$ | 3.0 |
| Martensite (S) | $1.33\times10^{-10}$ | 2.25 |
| Austenite (S) | $5.5\times10^{-12}$ | 3.25 |
| Cast Iron (S) | $5.5\times10^{-12}$ | 3.25 |
| Al-Alloy | $1.1\times10^{-11}$ | 3.89 |

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