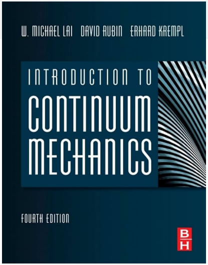
AS3020: Aerospace Structures
Module 3: Introduction to Elasticity

Instructor: Nidish Narayanaa Balaji

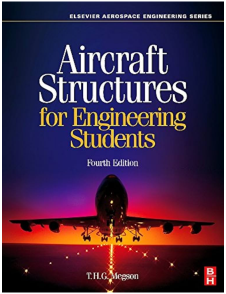
Dept. of Aerospace Engg., IIT-Madras, Chennai

August 19, 2025

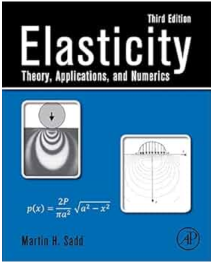
Table of Contents



Chapters 1-5 in Lai, Rubin, and Krempl (2010)



Chapters 1,2 in Megson (2013)

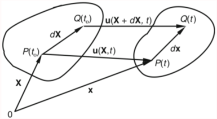


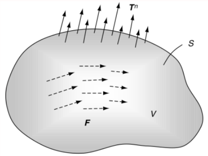
Chapters 1-5 in Sadd (2009)

We have to make a choice!

Red Pill

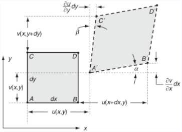
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

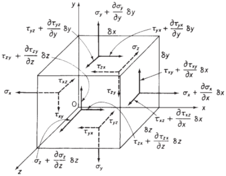




Blue Pill

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_y)$$

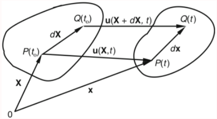


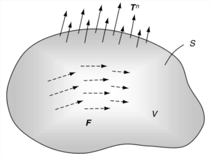


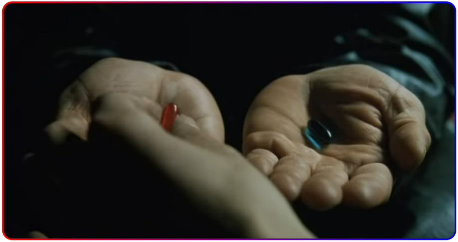
~~We have to~~ I make a choice!

Red Pill

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

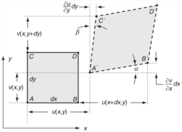


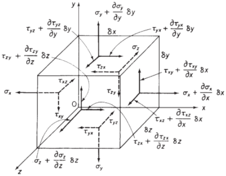




Blue Pill

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_y)$$





1.1. Indicial Notation I

1. Mathematical Rudiments

Einstein's Summation Convention: Dummy Indices

$$s = a_1 x_1 + a_2 x_2 + \cdots = \sum_{i=1}^{n} a_i x_i \to a_i x_i = a_k x_k = a_m x_m$$

Consider $\alpha = a_{ij} x_i x_j$, $\underline{v} = v_i \underline{e_i}$, $\underline{\underline{T}} = T_{ij} \underline{e_i} \underline{e_j}$

Free Indices

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}\right\} \implies y_i = a_{ij} x_j$$

Consider $T_{ij} = A_{im} A_{jm}$.

1.1. Indicial Notation II

1. Mathematical Rudiments

The Kronecker Delta

$$\delta_{ij} := \underline{e_i} \cdot \underline{e_j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $C_{ijkl} = \delta_{ik}\delta_{jl}$, $C_{ijkl} = \delta_{il}\delta_{jk}$.

The Levi-Civita Symbol

$$\epsilon_{ijk} := \underline{e}_i \cdot \underbrace{(\underline{e}_j \times \underline{e}_k)}_{\epsilon_{ijk}\hat{e}_i} = \begin{cases} 1 & \text{if } \{(i,j,k)\} \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & \text{if } \{(i,j,k)\} \in \{(3,2,1),(2,1,3),(1,3,2)\} \\ 0 & \text{otherwise} \end{cases}$$

Consider $\underline{a} \cdot (\underline{b} \times \underline{c})$, $\Delta \underline{\underline{F}}$.

1.1. Indicial Notation III

1. Mathematical Rudiments

Property: $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\epsilon_{ijk}\epsilon_{mnk} = (\epsilon_{ijk}\underline{e}_k) \cdot (\epsilon_{mnk}\underline{e}_k) = \left(\underline{e}_i \times \underline{e}_j\right) \cdot (\underline{e}_m \times \underline{e}_n)$$

$$\left(\underline{e}_i \times \underline{e}_j\right) \cdot (\underline{e}_m \times \underline{e}_n) = \begin{cases} 1, & \underline{e}_i \times \underline{e}_j = \underline{e}_m \times \underline{e}_n \\ -1, & \underline{e}_i \times \underline{e}_j = -\underline{e}_m \times \underline{e}_n = \underline{e}_n \times \underline{e}_m \\ 0, & \text{otherwise} \end{cases}$$

$$= \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

Consider $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d})$ (Lagrange's identity).

Derivative Notation

$$\frac{\partial u_i}{\partial x_j} := u_{i,j}$$

Consider $\nabla \underline{u}$, $\nabla \cdot \underline{u}$, $\nabla \times \underline{u}$, $\nabla \times \underline{Q}$

1.1. Indicial Notation IV

1. Mathematical Rudiments

Exercise

$$\nabla \underline{u}, \underbrace{\nabla \cdot (\nabla \underline{u})}_{\nabla^2 \underline{u}}, \nabla \cdot (\nabla \times \underline{u}), \nabla \times \nabla \times \underline{u}, \nabla \cdot \underline{\underline{\sigma}}$$

Vectors, Tensors

$$\underline{u} = u^i \hat{e}_i, \quad \underline{\underline{T}} = T^{ij} \hat{e}_i \hat{e}_j$$

Consider:

- Order of a tensor
- Vector-components as first order tensors
- The tensor product and 2nd order tensors
- Tensors as defining an operation
- Identity **tensors**
- Coordinate transformation
- "Notational abuse"
- Symmetric, antisymmetric tensors
- Antisymmetry as a cross product
- Representation of Eigen-decomposition
- Calculus: Gradient, Divergence, Laplacian, Curl, curvilinear coordinates

1.2. Matrix Algebra in Indicial Notation

Mathematical Rudiments

Indicial notation leads to some very nifty tricks while dealing with classical matrix algebra. Consider the following:

Determinant of a Matrix is Written as a scalar triple product of its columns or row vectors:

$$\underset{\approx}{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \rightarrow A_{ij}.$$

$$det(\underset{\approx}{A}) = \langle A_{1i}\underline{e_i} \times A_{2j}\underline{e_j}, A_{3k}\underline{e_k}\rangle = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$$
$$= \langle A_{i1}\underline{e_i} \times A_{j2}\underline{e_j}, A_{k3}\underline{e_k}\rangle = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

Rows(Columns) of the adjoint of a Matrix can be written as the components of the cross product of the remaining Column(Row) vectors

$$Adj(\underset{\approx}{A})_{1i} = \epsilon_{ijk}A_{j2}A_{k3}, \text{ and}$$
$$Adj(\underset{\approx}{A})_{i1} = \epsilon_{ijk}A_{2j}A_{3k}.$$

You should be able to verify easily that $Adj(\underset{\approx}{A})\underset{\approx}{A} = det(\underset{\approx}{A})\underset{\approx}{I}$.

The derivative of the determinant is simplified as

$$\frac{d}{dp}(det(\underset{\approx}{A})) = \frac{d}{dp}(\epsilon_{ijk}A_{1i}A_{2j}A_{3k}) = \epsilon_{ijk}\left(A'_{1i}A_{2j}A_{3k} + A_{1i}A'_{2j}A_{3k} + A_{1i}A_{2j}A'_{3k}\right)$$
$$= Adj(\underset{\approx}{A})_{i1}A'_{1i} + Adj(\underset{\approx}{A})_{j2}A'_{2j} + Adj(\underset{\approx}{A})_{k3}A'_{3k} = Adj(\underset{\approx}{A})_{ij}A'_{ji}$$
$$= trace(Adj(\underset{\approx}{A})_{ij}A'_{jk}) = trace(Adj(\underset{\approx}{A})\underset{\approx}{A}').$$

This will turn out to be quite an important result later on.

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Curvilinear Coordinates

- Scalar field ϕ gradient:
$$\delta\phi = \frac{\partial\phi}{\partial x_1}\delta x_1 + \frac{\partial\phi}{\partial x_2}\delta x_2 = \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta$$
- Polar bases
$$\underline{e}_r = C_\theta \underline{e}_1 + S_\theta \underline{e}_2 \implies \delta\underline{e}_r = \delta\theta\underline{e}_\theta$$
$$\underline{e}_\theta = -S_\theta \underline{e}_1 + C_\theta \underline{e}_2 \implies \delta\underline{e}_\theta = -\delta\theta\underline{e}_r$$
- Position vector
$$\delta\underline{r} = \delta r\underline{e}_r + r\delta\underline{e}_r = \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta$$
- For $\delta\phi = \nabla\phi\cdot\delta\underline{r}$,
$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta$$

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$
  Potential theory:
  $\int_{\partial\mathcal{D}} F_i dx_i = 0 \implies$
  - $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
  - $\underline{F} = \nabla\phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial\mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Diffe

- Scalar, vect
- Gradients,
- Divergence,
- Curvilinear divergence coordinate-

Stoke's Law as a Special Case of Gauss Divergence in 2D

$$
\begin{aligned}
\int_{\mathcal{S}} (\nabla \times \underline{v}) \cdot d\underline{S} &= \int_{\mathcal{S}} \epsilon_{ijk} v_{k,j} \hat{n}_i d|S| \\
&= \int_{\mathcal{S}} \left(\epsilon_{ijk} \hat{n}_i v_k\right)_{,j} d|S| \\
&= \int_{\partial\mathcal{S}} \epsilon_{ijk} \hat{n}_i v_k \hat{b}_j d|\ell| \\
&= \int_{\partial\mathcal{S}} \underbrace{\left(\epsilon_{ijk} \hat{n}_i \hat{b}_j\right)}_{\underline{\hat{n}} \times \underline{\hat{b}}} v_k d|\ell| \\
&= \int_{\partial\mathcal{S}} v_k t_k d|\ell| = \int_{\partial\mathcal{S}} \underline{v} \cdot d\underline{\ell}
\end{aligned}
$$

lus

$d\underline{x}$

$_{k,j}|_{\mathcal{D}} = 0$

$\underline{F} = \underline{0}$

heorem

$_{jk\ldots} dA_i$

$\underline{F} \cdot d\underline{x}$

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl
- Curvilinear coordinates: The divergence has to be *coordinate-independent*

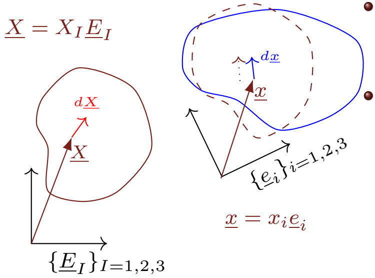
Integral Calculus

- The line integral: $\int \underline{F} \cdot d\underline{x}$
  Potential theory:
  $\int_{\partial\mathcal{D}} F_i dx_i = 0 \implies$
  - $F_i = \phi_{,i}$ and $\epsilon_{ijk} F_{k,j}|_{\mathcal{D}} = 0$
  - $\underline{F} = \nabla\phi$ and $\nabla \times \underline{F} = \underline{0}$
- **Gauss Divergence Theorem**
  $\int_{\mathcal{D}} P_{ijk\dots,i} d\mathcal{D} = \int_{\partial\mathcal{D}} P_{ijk\dots} dA_i$
- **Stoke's Law**:
  $\int_A (\nabla \times \underline{F}) \cdot d\underline{A} = \int_{\partial A} \underline{F} \cdot d\underline{x}$
- **Determinant of a Tensor**
  $\epsilon_{IJK} \Delta \underline{\underline{F}} = \epsilon_{ijk} F_{iI} F_{jJ} F_{kK}$
  - Related to volume change through transformation

2. Deformations and Strain

2.1. The Basic Premise

How to describe the change in shape **independently** of rigid body motions?



- The deformations are mapped as

$$\begin{aligned} \text{Lagrangian} \quad & x_i = x_i(\underline{X}) \\ \text{Eulerian} \quad & X_i = X_i(\underline{x}) \end{aligned}$$

- Under the **Lagrangian description** we have,

$$dx_i = \overbrace{\frac{\partial x_i}{\partial X_I}}^{F_{iI}} dX_I$$

$$\begin{aligned} \text{Length} \quad & ds^2 = dx_i dx_i = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I} \frac{\partial x_i}{\partial X_J} \right] dX_J \\ \text{Angle} \quad & ds_1 ds_2 \cos\theta = dx_i dx_j = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I} \frac{\partial x_j}{\partial X_J} \right] dX_J \end{aligned}$$

How does $d\underline{X}$ transform into $d\underline{x}$?

$$\underline{x} = \underline{X} + \underline{u}$$

2.2. Coordinate Transformation

2. Deformations and Strain

- A vector $\underline{v}$ is written as

$$\underline{v} = v_i \underline{e}_i,$$

and is defined as **a linear combination of the bases of its vector-space**.

- Suppose I have another coordinate system spanning the same vector-space, this comes with its own set of basis vectors $\{\underline{b}_i\}_{i=1,\ldots,n}$.
- If the vector represents a physical/geometrical measurement, it **can not change based on coordinate system**, i.e., it is coordinate invariant.
- So, the following equality must hold:

$$\underline{v} = v_i \underline{e}_i = \overline{v_i} \underline{b}_i,$$

with v_i and $\overline{v_i}$ being the **components of the vector** under the different coordinate systems.

2.2. Coordinate Transformation

2. Deformations and Strain

- Assuming that both $\{\underline{e}_i\}$ and $\{\underline{b}_i\}$ represent **orthogonal coordinate systems** (inner products $\langle \underline{e}_i, \underline{e}_j \rangle \equiv \langle \underline{b}_i, \underline{b}_j \rangle = \delta_{ij}$), we write down:

$$v_i = \langle \underline{v}, \underline{e}_i \rangle; \quad \overline{v_i} = \langle \underline{v}, \underline{b}_i \rangle.$$

- Evaluating $\overline{v_i}$ we obtain,

$$\overline{v_i} = \langle v_j \underline{e}_j, \underline{b}_i \rangle = \langle \underline{b}_i, \underline{e}_j \rangle v_j.$$

Denoting $\langle \underline{b}_i, \underline{e}_j \rangle = Q_{ij}$, we get our **component tranformation law for a vector**:

$$\boxed{\overline{v_i} = Q_{ij} v_j}$$

What about the basis vectors themselves?

How can I **combine** the $\underline{e}_i$'s to obtain the $\underline{b}_i$'s ?

- How **should I combine them** so that my vector is invariant?

2.2. Coordinate Transformation

2. Deformations and Strain

- Given the $\overline{v_i} = Q_{ij} v_j$, and the requirement $v_i \underline{e}_i = \overline{v_j} \underline{b}_j$, we write (after swapping $i \leftrightarrow j$ in LHS),

$$Q_{ji} v_i \underline{b}_j = v_i \underline{e}_i \implies Q_{ji} \underline{b}_j = \underline{e}_i \qquad \text{(multiply both sides by } (\mathbb{Q}^{-1})_{ik})$$

$$\overbrace{Q_{ji} (\mathbb{Q}^{-1})_{ik}}^{\delta_{jk}} \underline{b}_j = (\mathbb{Q}^{-1})_{ik} \underline{e}_i$$

$$\implies \underline{b}_i = (\mathbb{Q}^{-1})_{ji} \underline{e}_j$$

- Comparing the two, we have

$$\boxed{\overline{v_i} = Q_{ij} v_j} \qquad \boxed{\underline{b}_i = (\mathbb{Q}^{-1})_{ji} \underline{e}_j}$$

- This is a **necessary requirement** so that the vector **remains invariant**.

2.2. Coordinate Transformation: Array Notation

2. Deformations and Strain

- Now we introduce the Array Notation for vectors. **Let $\underline{v}$ be a vector.** The array of its components with respect to the basis $\{\underline{e}_i\}$ is written as,

$$\underset{\sim}{v} = \begin{bmatrix} \langle \underline{v}, \underline{e}_1 \rangle \\ \langle \underline{v}, \underline{e}_2 \rangle \\ \vdots \end{bmatrix} \qquad \text{(similarly for } \underset{\sim}{\overline{v}}\text{)}.$$

- We also define the **array of coordinate vectors** as

$$\underset{\sim}{\underline{e}} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \vdots \end{bmatrix}; \qquad \underset{\sim}{\underline{b}} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \end{bmatrix}.$$

- Under this notation we have,

$$\boxed{\underset{\sim}{\overline{v}} = \mathbb{Q}\underset{\sim}{v}} \quad \text{and} \quad \boxed{\underset{\sim}{\underline{b}} = \mathbb{Q}^{-T}\underset{\sim}{\underline{e}}}.$$

$\underset{\sim}{\overline{v}}$ "contra-varies" w.r.t. $\underset{\sim}{v}$, **in comparison with** how $\underset{\sim}{\underline{b}}$ and $\underset{\sim}{\underline{e}}$ are related. $\implies \underset{\sim}{\tilde{v}}$ are the **contravariant components** of $\underline{v}$.

2.2. Coordinate Transformation: Tensors

2. Deformations and Strain

- We will define a (2nd order) tensor as a **linear combination of basis-dyads**:

$$\underline{\underline{T}} = T_{ij}\underline{e}_i\underline{e}_j = \overline{T}_{ij}\underline{b}_i\underline{b}_j,$$

where **we have required $\underline{\underline{T}}$ to be invariant** under coordinate change.

- Using a **double-contraction** operation, we write down the components of $\overline{T}_{ij}$ as,

$$\overline{T}_{ij} = T_{mn}\underbrace{\langle \underline{b}_i, \underline{e}_m\rangle}_{Q_{im}}\overbrace{\langle \underline{b}_j, \underline{e}_n\rangle}^{Q^{jn}}$$
$$= Q_{im}T_{mn}Q_{jn}.$$

- In array notation we write the components as,

$$\boxed{\overline{\mathbb{T}} = \mathbb{Q}\mathbb{T}\mathbb{Q}^T}.$$

For a tensor to be invariant, its components have to transform in this fashion.

2.2. Coordinate Transformation: Summary

2. Deformations and Strain

Supposing I specify a basis change by

$$\underset{\sim}{\underline{b}} = \mathbb{Q}^{-T} \underset{\sim}{\underline{e}},$$

- for a vector $\underline{v} = \underset{\sim}{v}^T \underset{\sim}{\underline{e}}$ to be invariant, its components have to transform as

$$\underset{\sim}{\overline{v}} = \mathbb{Q} \underset{\sim}{v}.$$

- for a tensor $\underline{\underline{T}} = \mathbb{T} \underset{\sim}{\underline{e}} \otimes \underset{\sim}{\underline{e}}$ to be invariant, its components have to transform as

$$\overline{\mathbb{T}} = \mathbb{Q}\mathbb{T}\mathbb{Q}^T$$

- If it transforms in any other fashion, **then invariance is not guaranteed**, or in other words, **the quantity is not objective**.

2.2. Coordinate Transformation: Relationship to Gradients

2. Deformations and Strain

We will now establish a relationship between coordinate transformation and **component-gradients**.

- Consider an infinitesimal line vector $d\underset{\sim}{x} = dx_i \underline{e}_i = d\overline{x}_i \underline{b}_i$.
- It is obvious that the components $d\underset{\sim}{\overline{x}}$ have to be related to the components $d\underset{\sim}{x}$. So we write

$$d\overline{x}_i = \frac{\partial \overline{x}_i}{\partial x_j} dx_j \tag{1}$$

- By invariance requirements, we have

$$d\overline{x}_i = Q_{ij} dx_j. \tag{2}$$

- Comparing eq. (1) and eq. (2) we obtain,

$$\boxed{Q_{ij} = \frac{\partial \overline{x}_i}{\partial x_j}} \quad \text{or} \quad \boxed{\mathbb{Q} = grad\left(\underset{\sim}{\overline{x}}\right)}$$

$grad(\cdot)$ operator $\Longrightarrow$ gradient operation

2.2. Coordinate Transformation: The Deformation Gradient

2. Deformations and Strain

- The **components of the deformation gradient** are written as

$$F_{iI} = \frac{\partial x_i}{\partial X_I}.$$

- Under coordinate change we have,

$$\begin{aligned}\overline{F}_{iI} &= \frac{\partial \overline{x}_i}{\partial x_j}\frac{\partial x_j}{\partial X_J}\frac{\partial X_J}{\partial \overline{X}_I}\\ &= Q^{(x)}_{ij} F_{jJ} (\mathbb{Q}^{(X)^{-1}})_{JI} \implies \boxed{\overline{\mathbb{F}} = \mathbb{Q}^{(x)}\mathbb{F}\mathbb{Q}^{(X)^{-1}}}.\end{aligned}$$

This is transforming quite unlike a tensor for 2 reasons

1. $\mathbb{Q}^{(x)}$ and $\mathbb{Q}^{(X)}$ **need not necessarily be the same** (we are free to choose measurement coordinates at each instant)
2. $\mathbb{Q}^{-1} = \mathbb{Q}^T$ only for orthonormal coordinate systems (cartesian, for eg.). For non-orthonormal bases, this is not so.

2.2. Coordinate Transformation: The Cauchy Deformation Tensor

2. Deformations and Strain

- Now we consider $\mathbb{C} = \mathbb{F}^T\mathbb{F}$. Under coordinate change this becomes,

$$\begin{aligned}\overline{\mathbb{C}} = \overline{\mathbb{F}}^T\overline{\mathbb{F}} &= \left(\mathbb{Q}^{(x)}\mathbb{F}\mathbb{Q}^{(X)^{-1}}\right)^T\left(\mathbb{Q}^{(x)}\mathbb{F}\mathbb{Q}^{(X)^{-1}}\right)\\ &= \mathbb{Q}^{(X)^{-T}}\mathbb{F}^T\mathbb{Q}^{(x)^T}\mathbb{Q}^{(x)}\mathbb{F}\mathbb{Q}^{(X)^{-1}}\end{aligned}$$

- Suppose we choose to stick with **coordinate systems with orthonormal bases**, $\mathbb{Q}^{-1} = \mathbb{Q}^T$ (for both (x) and (X)). Hereby the components matrix $\mathbb{C}$ reduces to

$$\boxed{\mathbb{C} = \mathbb{Q}^{(X)}\mathbb{F}^T\mathbb{F}\mathbb{Q}^{(X)^T}}$$

Unlike the deformation gradient...

...this is transforming like a tensor's components!
So I can define the **Cauchy deformation tensor** as: $\underline{\underline{C}} = C_{IJ}\underline{E}_I\underline{E}_J$

2.3. The Strain Tensor

2. Deformations and Strain

- We are now ready to define the strain tensor based on length change. We wrote,

$$\begin{aligned} ds^2 - dS^2 &= dX_I \left(F_{iI} F_{jJ} - \delta_{IJ} \right) dX_J \\ &= d\underset{\sim}{X}^T \left[\mathbb{F}^T \mathbb{F} - \mathbb{I} \right] d\underset{\sim}{X} = d\underset{\sim}{X}^T \left[\mathbb{C} - \mathbb{I} \right] d\underset{\sim}{X}. \end{aligned}$$

- For small changes in length, $ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$.
- Representing the elongation as a fraction of the total length we write $(ds - dS) = \epsilon dS$. Using this we have,

$$2dS^2 \epsilon = d\underset{\sim}{X}^T [\mathbb{C} - \mathbb{I}] d\underset{\sim}{X} \implies 2d\underset{\sim}{X}^T d\underset{\sim}{X} \epsilon = d\underset{\sim}{X}^T [\mathbb{C} - \mathbb{I}] d\underset{\sim}{X}.$$

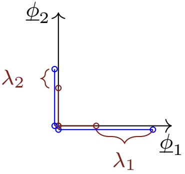
  Here the single factor ϵ represents what the matrix $\mathbb{E} = \frac{1}{2} \left[\mathbb{C} - \mathbb{I} \right]$ is doing in the **bi-linear form** $d\underset{\sim}{X}^T \mathbb{E} d\underset{\sim}{X}$.

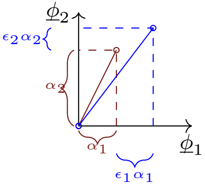
The matrix **E** represents the components of the **Strain Tensor**.

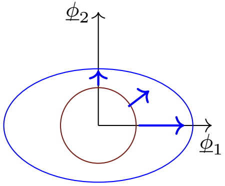
2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

- Consider the operation $\mathbb{E}\underset{\sim}{u}$. Say, $\underset{\sim}{v} = \mathbb{E}\underset{\sim}{u}$.
  $\underset{\sim}{v}$ represents the **components of a vector** which can be <u>arbitrarily oriented</u> w.r.t. $\underset{\sim}{u}$.
- Consider some unit vector $\underset{\sim}{\phi}$ such that $\mathbb{E}\underset{\sim}{\phi} = \lambda\underset{\sim}{\phi}$.
  The operation of the matrix $\mathbb{E}$ leads to perfect stretching by a factor of λ.
- The pair $(\lambda, \underset{\sim}{\phi})$ are known as an **eigenpair** of $\mathbb{E}$ $\underset{\sim}{\phi}$ represents a <u>principal direction</u>.
- For 3D mechanics, we have 3 principal directions.
  Consider the 2D case below:





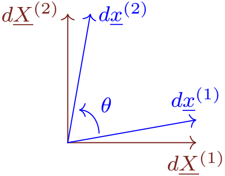


2.3. The Strain Tensor: Infinitesimal Case

2. Deformations and Strain

- $d\underset{\sim}{X}^T \mathbb{E} d\underset{\sim}{X}$ represents elongation/shortening of length **without regard to orientation changes**.
- For considering orientation change, it is **not enough just to look at a single line-segment**.
- Let us consider 2 line-vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$ that are **perpendicular in the undeformed condition** $\implies \langle d\underline{X}^{(1)}, d\underline{X}^{(2)} \rangle = d\underset{\sim}{X}^{(1)^T} d\underset{\sim}{X}^{(2)} = 0$.
- In the deformed condition, the inner product is $\langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle = d\underset{\sim}{X}^{(1)^T} \mathbb{C} d\underset{\sim}{X}^{(2)} = d\underset{\sim}{X}^{(1)^T} 2\mathbb{E} d\underset{\sim}{X}^{(2)}$.
- For small angle changes, the LHS simplifies as,

$$\begin{aligned} \langle d\underline{x}^{(1)}, d\underline{x}^{(2)} \rangle = |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \cos\theta &\approx |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \left(0 + (\theta - \frac{\pi}{2})(-1) + \dots\right) \\ &= |d\underline{x}^{(1)}||d\underline{x}^{(2)}| \underbrace{(\frac{\pi}{2} - \theta)}_{\gamma} \end{aligned}$$



2.3. The Strain Tensor: Shear Strain

2. Deformations and Strain

- Consider $d\underline{X}^{(1)} = |d\underline{X}^{(1)}|\underline{e}_1$, $d\underline{X}^{(2)} = |d\underline{X}^{(2)}|\underline{e}_2$.
  Then we have,

$$d\underset{\sim}{X}^{(1)^T} \mathbb{E} d\underset{\sim}{X}^{(2)} = |d\underline{X}^{(1)}||d\underline{X}^{(2)}|E_{12},$$

  — i.e., the off-diagonal component E_{12}.

- So the complete equality is written as,

$$|d\underline{x}^{(1)}||d\underline{x}^{(2)}|\gamma = |d\underline{X}^{(1)}||d\underline{X}^{(2)}|2E_{12}.$$

- Under the condition of no elongation (pure shear), the off-diagonal components measure the angle-change.
- We will interpret it as being under the condition of small elongation.

2.3. The Strain Tensor: In terms of displacement

2. Deformations and Strain

Let us now express strain in terms of the displacement field $\underline{u}(\underline{X})$.

- We have $x_i = X_i + u_i$. So the deformation gradient is written as,

$$F_{iI} = \frac{\partial x_i}{\partial X_I} = \delta_{iI} + u_{i,I}.$$

- Cauchy deformation tensor is written as (with components $\mathbb{C} = \mathbb{F}^T\mathbb{F}$),

$$C_{IJ} = F_{iI}F_{iJ} = \delta_{IJ} + u_{I,J} + u_{J,I} + u_{i,I}u_{i,J}.$$

- From this, the strain tensor is written as (with components $\mathbb{E} = \frac{1}{2}(\mathbb{C} - \mathbb{I})$)

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \underbrace{\frac{\partial u_i}{\partial X_I}\frac{\partial u_i}{\partial X_J}}_{\text{ignored for small strain}}\right)$$

- **Infinitesimal Strain Tensor:** $E_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I})$.

2.3. The Strain Tensor: Volume Change

2. Deformations and Strain

- Consider three arbitrarily oriented vectors $d\underline{X}^{(1)}$, $d\underline{X}^{(2)}$, $d\underline{X}^{(3)}$ in the undeformed configuration. The volume that they describe is given by

$$dV = \epsilon_{IJK} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}.$$

- Upon deformation, using the same notation as above, the volume **becomes**

$$dv = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}.$$

Using the deformation gradient to write this out ($d\underset{\sim}{x} = \mathbb{F} d\underset{\sim}{X}$), we have

$$dv = \underbrace{\epsilon_{ijk} F_{iI} F_{jJ} F_{kK}} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}$$

- We have previously seen that $\epsilon_{ijk} F_{iI} F_{jJ} F_{kK} = \epsilon_{IJK} det(\mathbb{F})$. Substituting this in the above we get,

$$dv = \epsilon_{IJK} det(\mathbb{F}) dX_I^{(1)} dX_J^{(2)} dX_K^{(3)} = det(\mathbb{F}) dV.$$

- $J := det(\mathbb{F})$ is known as the *Jacobi determinant.* $\boxed{dv = JdV}$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- For the infinitesimal case, the deformation gradient component matrix is expressed as

$$\mathbb{F} = \mathbb{I} + \epsilon \nabla u,$$

where $\epsilon > 0$ is **some small number** ($\epsilon \ll 1$).

- Since ϵ is small, we will try to expand out J as a Taylor series in ϵ about $\epsilon = 0$:

$$J(\epsilon) = J(\epsilon = 0) + \epsilon \left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} + \mathcal{O}(\epsilon^2).$$

Derivative of Determinant

$$\frac{d}{dp}\left(det(\mathbb{M})\right) = trace\left(Adj(\mathbb{M})\frac{d\mathbb{M}}{dp}\right)$$

For invertible $\mathbb{M}$, $Adj(\mathbb{M}) = J\mathbb{M}^{-1}$.

- This simplifies as,

$$J(\epsilon) = det(\mathbb{I}) + \epsilon\left(J(\epsilon = 0) trace\left(\mathbb{I}^{-1}\nabla u\right)\right) + \mathcal{O}(\epsilon^2) \approx 1 + \epsilon tr(\nabla u)$$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- Undeformed volume is $\boxed{dV}$, deformed volume is $\boxed{dv = JdV}$. So **relative change in volume is**

$$\frac{dv - dV}{dV} = J - 1.$$

- For the infinitesmial case $J \approx 1 + tr(\nabla u)$ (we have set $u \to \epsilon u$ here). Substituting, we get

$$\frac{dv - dV}{dV} = tr(\nabla u) = u_{I,I} = E_{II} = tr(\mathbb{E}).$$

- So the **trace of the strain tensor** is the relative volume change.

In Summary we have, for the strain tensor,

- Each diagonal element corresponds to **stretching/compressing**,
- Off-diagonal elements correspond to **shearing**,
- Trace (sum of diagonal elements) corresponds to **volume change**.

Summary
2. Deformations and Strain

- We have defined the deformation gradient $\mathbb{F}$ and the strain tensor $\underline{\underline{E}}$.
- **Notice:** Under no deformation, if you just changed the coordinate frame of observation, $\mathbb{F}$ will change, **but** $\underline{\underline{E}}$ **will not**.

Rigid Body Motion

$$\underset{\sim}{x} = \underset{\sim}{c} + \mathbb{R}(\underset{\sim}{X} - \underset{\sim}{X}_0)$$

- What is the deformation gradient here?
- What is the **infinitesimal strain tensor** here?
- What is the **finite strain tensor** here?

- What should the material **respond to**? What is the quantity that the **material wants to resist**?

2.4. Strain Compatibility

2. Deformations and Strain

Necessary Reading

Read Section 1.10 in Megson (2013)

- Since strains are defined **based on the displacement field**, the different strain components are related.
- For the infinitesimal case we have: $2E_{ij} = u_{i,j} + u_{j,i}$.
  We want to manipulate this such that we get an **equality fully expressed in the strains alone.**
- Differentiating by X_k and premultiplying by ϵ_{jkm} we have,

$$2\epsilon_{jkm}E_{ij,m} = \cancelto{0}{\epsilon_{jkm}u_{i,jk}} + \epsilon_{jkm}u_{j,ik} \rightarrow \boxed{\text{free indices: } i, m}$$

- We differentiate thisa by X_l and premultiply by ϵ_{iln} to get:

$$2\epsilon_{iln}\epsilon_{jkm}E_{ij,mn} = \cancelto{0}{\epsilon_{jkm}\epsilon_{iln}u_{j,ikl}} \rightarrow \boxed{\text{free indices: } k, l}$$

2.4. Strain Compatibility

2. Deformations and Strain

- The compatibility equation $\boxed{\epsilon_{mjk}\epsilon_{nil}E_{ij,mn} = 0}$ represents a 3×3 system of 9 equations.
- We have **two symmetries**: $E_{ij} = E_{ji}$ (strain tensor symmetry), and $E_{ij,kl} = E_{ij,lk}$ (strain continuously differentiable).
  Applying this can convince us that the equation is also symmetric. So we have $\frac{3(3+1)}{2} = 6$ **unique equations**.
- In component notation, these can be written out as,

$$\begin{aligned}
&(k,l)=(1,1) \quad E_{22,33} + E_{33,22} = 2E_{23,23}, \quad (k,l)=(1,3) \quad E_{22,13} + E_{13,22} = E_{12,23} + E_{23,12} \\
&(k,l)=(2,2) \quad E_{33,11} + E_{11,33} = 2E_{13,13}, \quad (k,l)=(1,2) \quad E_{33,12} + E_{12,33} = E_{13,23} + E_{23,13} \\
&(k,l)=(3,3) \quad E_{11,22} + E_{22,11} = 2E_{12,12}, \quad (k,l)=(2,3) \quad E_{11,23} + E_{23,11} = E_{12,13} + E_{13,12}
\end{aligned}$$

The strains have to satisfy these conditions for them to "have been generated" by a continuously differentiable displacement field.

3. Stress and Equilibrium

Force is a vector. Area is a vector. What is **pressure** (F/A)?

- Consider a small area ΔA in a cut-section of an elastic body as shown. The **traction vector** $\underline{t}$ is the limiting force

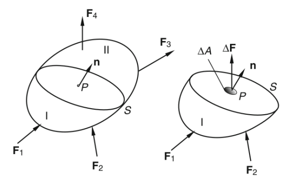


Figure from Lai, Rubin, and Krempl 2010

$$\underline{t} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}.$$

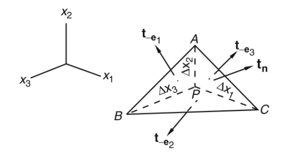


Figure from Lai, Rubin, and Krempl 2010

- By basic force-balance arguments, we can argue that the relationship between the traction vector and the normal vector to the chosen area **has to be linear**.

$$\boxed{t_i = \sigma_{ij} n_j}.$$

Cauchy Stress Tensor: $\sigma_{ij} \underline{e}_i \underline{e}_j$

Force Equilibrium

3. Stress and Equilibrium

- Consider the forces on a small volume dv in the **deformed configuration** (denoted Ω_d):

  Body loads $\int_{\Omega_d} f_i(\underline{x}) dv$

  Surface tractions $\int_{\partial\Omega_d} t_i d|a|$
- Static equilibrium is written as

$$\int_{\partial\Omega_d} \sigma_{ij} da_j + \int_{\Omega_d} f_i dv = 0.$$

- Applying Gauss divergence, this simplifies to,

$$\int_{\Omega_d} \sigma_{ij,j} + f_i dv = 0 \implies \boxed{\sigma_{ij,j} + f_i = 0}.$$

- This is the **static equilibrium equation**.

Moment Equilibrium

3. Stress and Equilibrium

- We next consider the balance of the moments of forces on the same differential element.

$$\int_{\partial\Omega_d} \underbrace{\epsilon_{ijk} x_j \sigma_{kl} da_l}_{\underline{x}\times\underline{t}d|a|} + \int_{\Omega_d} \epsilon_{ijk} x_j f_k dv = 0.$$

- Applying Gauss divergence again we get,

$$\int_{\Omega_d} \epsilon_{ijk}((x_j\sigma_{kl})_{,l} + x_j f_k) dv = \int_{\Omega_d} \epsilon_{ijk}\left(\delta_{jl}\sigma_{kl} + x_j \cancel{(\sigma_{kl,l} + f_k)}\right) dv = 0$$

$$\implies \boxed{\epsilon_{ijk}\sigma_{jk} = 0}$$

which is an assertion of **symmetry of the stress tensor**.

- Note that we have <u>assumed the absense of body moments</u> here.

3.1. Stress Work Done

- Let us now consider the work done by the stress. For convenience, we start with the **rate of work done**: force×velocity .
- On the infinitesimal element we have,

$$\frac{dU}{dt} = \int_{\partial\Omega_d} \sigma_{ij}\dot{u}_i da_j + \int_{\Omega_d} f_i \dot{u}_i dv.$$

- Application of Gauss divergence leads to,

$$\frac{dU}{dt} = \int_{\Omega_d} \left(\sigma_{ij}\dot{u}_i\right)_{,j} + f_i\dot{u}_i dv = \int_{\Omega_d} \sigma_{ij}\dot{u}_{i,j} + \dot{u}_i \cancel{(\sigma_{ij,j} + f_i)} dv$$

$$\implies \frac{dU}{dt} = \int_{\Omega_d} \sigma_{ij} \underbrace{\frac{\partial \dot{u}_i}{\partial X_I}}_{\dot{F}_{iI}} \underbrace{\frac{\partial X_I}{\partial x_j}}_{(\mathbb{F}^{-1})_{Ij}} dv$$

- The **power density is written as**,

$$\boxed{\frac{d\mathcal{U}}{dt} = \sigma_{ij} (\mathbb{F}^{-1})_{Ij} \dot{F}_{iI}}.$$

3.1. Stress Work Done: Non-Dissipative Solid

3. Stress and Equilibrium

- For a general non-dissipative solid, the work done must be path-independent, i.e., the contents of the energy integral must be an **exact differential** of the conserved quantity (stress/strain energy).
- Here we have,

$$\int_{\Omega_d} \frac{d\mathcal{U}}{dt} dv = \int_{\Omega_d} \frac{\partial \mathcal{U}}{\partial F_{iI}} \dot{F}_{iI} dv = \int_{\Omega_d} \sigma_{ij} (\mathbb{F}^{-1})_{Ij} \dot{F}_{iI} dv.$$

- It must be noted that the domain of integration, Ω_d is also **deformation dependent**, making this inconvenient. So we **map everything back to the undeformed reference (denoted Ω)**:

$$\int_{\Omega} \frac{\partial \mathcal{U}}{\partial F_{iI}} \dot{F}_{iI} \underbrace{det(\mathbb{F}) dV}_{dv} = \int_{\Omega} \sigma_{ij} (\mathbb{F}^{-1})_{Ij} \dot{F}_{iI} det(\mathbb{F}) dV$$

$$\implies \boxed{\sigma_{ij} = \frac{1}{det(\mathbb{F})} \frac{\partial \mathcal{U}}{\partial F_{iI}} F_{jI}}, \qquad \boxed{\underset{\approx}{\sigma} = \frac{1}{det(\mathbb{F})} \frac{\partial \mathcal{U}}{\partial \mathbb{F}} \mathbb{F}^T}.$$

3.1. Stress Work Done: Non-Dissipative Solid under infinitesimal strain

3. Stress and Equilibrium

- For the infinitesimal strain case, it can be shown that the above expression simplifies to,

$$\boxed{\sigma_{IJ} = \frac{\partial \mathcal{U}}{\partial E_{IJ}}}.$$

- Intuitively, under this condition, the deformed and undeformed coordinates are **almost the same**. Mathematically, this can be worked out by using a perturbative formalism by setting $\underline{u} \rightarrow \epsilon \underline{u}$.

4. Constitutive Relationships

- We have developed tensor-representations of both the stress, $\underline{\underline{\sigma}} = \sigma_{ij}\underline{e}_i\underline{e}_j$ and strain, $\underline{\underline{E}} = E_{ij}\underline{e}_i\underline{e}_j$. **We are now interested in relating the components of the two.**
- The most general **linear** relationship that one can assume is

$$\sigma_{ij} = C_{ijkl}E_{kl}.$$

- If the system is **non-dissipative**, then the stress must be expressible as $\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial E_{ij}}$. So,

$$\frac{\partial^2 \mathcal{U}}{\partial E_{ij}\partial E_{kl}} = C_{ijkl}.$$

- Since we expect a smooth energy density, the indices (i, j) and (k, l) must be swappable. This represents the first symmetry property of C_{ijkl} $(i, j \leftrightarrow k, l)$.
- Since stress and strain are also symmetric, the following index-swaps must be permissible: $i \leftrightarrow j$, $k \leftrightarrow l$.

Simplification Arguments

4. Constitutive Relationships

- In summary we have the following roadmap for simplification:

General Case	C_{ijkl}	$3 \times 3 \times 3 \times 3 = 81$ terms
Stress-Strain Symmetry	$i \leftrightarrow j$, $k \leftrightarrow l$	$\frac{3(3+1)}{2} \times \frac{3(3+1)}{2} = 36$ terms
Non-dissipativity, smoothness	$(i,j) \leftrightarrow (k,l)$	$\frac{6(6+1)}{2} = 21$ terms

- Suppose the **material is isotropic**, then the components C_{ijkl} are invariant under coordinate transformations. This means that it must be composed of $\delta_{..}$ symbols.
- Under symmetry, we have 3 unique combinations:

$$\delta_{ij}\delta_{kl}, \quad \delta_{ik}\delta_{jl}, \quad \delta_{il}\delta_{jk},$$

and we write:

$$C_{ijkl} = \alpha_1 \delta_{ij}\delta_{kl} + \alpha_2 \delta_{ik}\delta_{jl} + \alpha_3 \delta_{il}\delta_{jk}.$$

- Applying this to the stress-strain relationship, we get:

$$\sigma_{ij} = \alpha_1 \delta_{ij} E_{kk} + \alpha_2 E_{ij} + \alpha_3 E_{ji} \implies \boxed{\sigma_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}}.$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Consider a 2D case with $\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$.
- Consider a plane section with normal $\underset{\sim}{\hat{n}} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}^T$. The perpendicular is denoted $\underset{\sim}{\hat{s}} = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}^T$.
- The traction vector is given by $\underset{\sim}{t} = \underset{\approx}{\sigma}\underset{\sim}{\hat{n}}$:

$$\underset{\sim}{t} = \begin{bmatrix} \sigma_{11}\cos\theta + \sigma_{12}\sin\theta \\ \sigma_{12}\cos\theta + \sigma_{22}\sin\theta \end{bmatrix}.$$

- This is resolved along the $(\underset{\sim}{\hat{n}}, \underset{\sim}{\hat{s}})$ directions by the coordinate transformation,

$$\begin{bmatrix} \sigma_n \\ \tau_s \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\hat{n}}^T \\ \underset{\sim}{\hat{s}}^T \end{bmatrix} \underset{\sim}{t} = \begin{bmatrix} \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta \\ -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Now we consider two infinitesimal lines initially oriented along $\underline{\hat{n}}$ and $\underline{\hat{t}}$ ($dS\underline{\hat{n}}$, $dS\underline{\hat{t}}$).
- $dS\hat{n}$ experiences the elongation,

$$\frac{ds - dS}{dS} = \underset{\sim}{\hat{n}}^T \mathbb{E} \underset{\sim}{\hat{n}} = \epsilon_\ell.$$

- The shear strain between them is,

$$\gamma_s = 2\underset{\sim}{\hat{t}}^T \mathbb{E} \underset{\sim}{\hat{n}}.$$

- Simplifying, we get

$$\begin{bmatrix} \epsilon_\ell \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \frac{E_{11}+E_{22}}{2} + \frac{E_{11}-E_{22}}{2}\cos 2\theta + E_{12}\sin 2\theta \\ -(E_{11} - E_{22})\sin 2\theta + 2E_{12}\cos 2\theta \end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- When no shear load/response is observed, these reduce to,

$$\boxed{\begin{bmatrix}\sigma_n\\ \tau_s\end{bmatrix} = \begin{bmatrix}\frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta\\ -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta\end{bmatrix}}, \quad \boxed{\begin{bmatrix}\epsilon_\ell\\ \gamma_s\end{bmatrix} = \begin{bmatrix}\frac{E_{11}+E_{22}}{2} + \frac{E_{11}-E_{22}}{2}\cos 2\theta\\ -(E_{11}-E_{22})\sin 2\theta\end{bmatrix}}.$$

- For a linear-elastic material, **causal links** may be made between $\sigma_n \leftrightarrow \epsilon_\ell$ and $\tau_s \leftrightarrow \gamma_s$.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- From basic arguments one can motivate

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}).$$

- For the 2D case under pure tension,

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad E_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22}.$$

- For some section oriented by angle θ we have,

$$\gamma_s(\theta) = -(E_{11} - E_{22})\sin 2\theta = -\frac{1+\nu}{E}\underbrace{(\sigma_{11} - \sigma_{22})\sin 2\theta}_{2\tau_s},$$

which implies, $\boxed{\gamma_s(\theta) = 2\frac{1+\nu}{E}\tau_s}$.

- E: Young's Modulus, ν: Poisson's Ratio, and $G = \frac{E}{2(1+\nu)}$: Shear Modulus.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- We have also spoken about volume change. In terms of strains this is,

$$\begin{aligned}\frac{dv - dV}{dv} &= E_{11} + E_{22} + E_{33}\\ &= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}).\end{aligned}$$

- In other words we have $E_{ii} = \kappa\sigma_{ii}$, where $\boxed{\kappa = \frac{1-2\nu}{E}}$, the bulk modulus.
- From physical arguments, it is clear that $\kappa > 0$, which implies $\nu < 0.5$, which presents an **upper bound for the Poisson's ratio**.
- The shear modulus must also be positive. So we have $\frac{E}{2(1+\nu)} > 0$, which implies $\boxed{\nu > -1}$, which presents a **lower bound for the Poisson's ratio**.
- In summary we have, $\boxed{\nu \in (-1, 0.5)}$, $\boxed{E > 0}$.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- In tensor notation, this can be written as,

$$E_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right].$$

5. 2D Problems

- In 2D, the governing equations can be written as,

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$
$$\sigma_{12,1} + \sigma_{22,2} + f_2 = 0.$$

- Differentiation the first by X_1 and the second by X_2 leads to

$$\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} + f_{1,1} + f_{2,2} = 0.$$

- Strain Compatibility equations in 2D reads:

$$2E_{12,12} = E_{11,22} + E_{22,11}$$

- We, however, need compatibility in terms of stresses, not strains. Now we formalize the notion of two dimensions:

  Plane Stress $\sigma_{33} = 0$
  Plane Strain $E_{33} = 0$

The "Plane Stress" Case

5. 2D Problems

- Here, we assume $\sigma_{33} = 0$ (but $E_{33} \neq 0$ in general). So the stress-strain relationships are,

$$E_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad E_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11}$$
$$2E_{12} = 2\frac{1+\nu}{E}\sigma_{12}, \quad E_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

- Substituting this into the compatibility equations we get,

$$\implies \frac{2(1+\nu)}{E}\sigma_{12,12} = \frac{1}{E}\left((\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + \sigma_{22})_{,11}\right)$$
$$= \frac{1}{E}\left((\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22})\right)$$

- Combining the two we get,

$$\frac{\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}}{1+\nu} + f_{1,1} + f_{2,2} = 0 => \boxed{\sigma_{ii,jj} + (1+\nu)f_{i,i} = 0}.$$

The "Plane Strain" Case

5. 2D Problems

- Here, we assume $E_{33} = 0$ ($\sigma_{33} \neq 0$ in general). So the stress-strain relationships are simplified as,

$$E_{33} = \frac{\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})}{E} = 0 \implies \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}),$$

$$\implies E_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}) = \frac{1-\nu^2}{E}\sigma_{11} - \frac{\nu(1+\nu)}{E}\sigma_{22}$$

$$\implies E_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}(\sigma_{11} + \sigma_{33}) = \frac{1-\nu^2}{E}\sigma_{22} - \frac{\nu(1+\nu)}{E}\sigma_{11}$$

- Substituting this into the compatibility equations we get,

$$\implies \frac{2(1+\nu)}{E}\sigma_{12,12} = \frac{1+\nu}{E}\Big(((1-\nu)\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + (1-\nu)\sigma_{22})_{,11}\Big)$$

$$= \frac{1+\nu}{E}\left((1-\nu)(\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22})\right)$$

- Combining the two we get,

$$(1-\nu)(\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}) + f_{1,1} + f_{2,2} = 0 => \boxed{\sigma_{ii,jj} + \frac{1}{1-\nu}f_{i,i} = 0}.$$

5.1. The Airy's Stress Function

5. 2D Problems

- We can now combine both the **governing equations** and the **compatibility equations**, so we can write out the solution **fully in terms of stress only**.
- For the homogeneous case ($f_i = 0$), we have (for both plane stress and plane strain),

$$\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right)(\sigma_{11} + \sigma_{22}) = 0. \tag{3}$$

- We introduce the Airy's Stress function ϕ that simplifies the system of two PDE's into a scalar PDE by the substitutions:

$$\sigma_{11} := \frac{\partial^2 \phi}{\partial X_2^2}, \quad \sigma_{22} := \frac{\partial^2 \phi}{\partial X_1^2}, \quad \sigma_{12} := -\frac{\partial \phi}{\partial X_1 \partial X_2}.$$

(it is easily verified that this satisfies the governing equations $\sigma_{ij,j} = 0$ by definition)

- Substitution into eq. (3) leads to

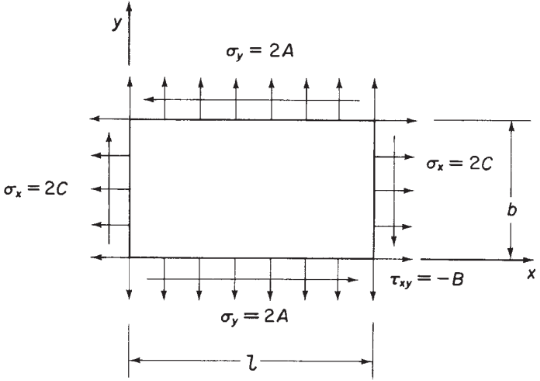
$$\phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right)^2 \phi = 0, \qquad \boxed{\nabla^4 \phi = 0},$$

sometimes known as the **Biharmonic Equation**.

5.1. The Airy's Stress Function: Tutorial

5. 2D Problems

- The Airy stress function can be used to solve problems with boundary loads. Consider this simple example from your textbook:

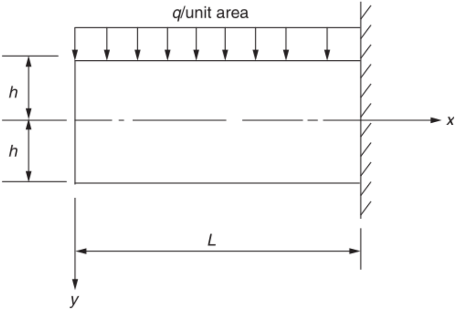


- **Airy Stress Function**: $\phi = Ax^2 + Bxy + Cy^2$

5.1. The Airy's Stress Function: Tutorial

5. 2D Problems

- Consider this second example from your text book (example 2.3):



Boundary Conditions

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad y = h$$
$$\sigma_{11} = \sigma_{12} = 0, \quad y = h$$
$$\sigma_{22} = -q, \quad y = -h$$
$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad x = 0$$
$$\int_{-h}^{h} y\sigma_{11} dy = 0, \quad x = 0$$

with a candidate Airy stress function $\phi(x, y) = Ax^2 + Bx^2y + Cy^3 + D(5x^2y^3 - y^5)$.

5.1. The Airy's Stress Function

5. 2D Problems

- It may be the case that the Airy stress function doesn't meet all the boundary conditions. In this case we find a stress function that **approximately satisfies the BCs in some sense**.
- So is this completely useless? No.

St. Venant's Principle (rephrased as in Lai, Rubin, and Krempl 2010)

*If **some distribution of forces acting on a portion of the surface of a body** is replaced by a **different distribution** of forces acting on the **same portion of the body**, then the **effects** of the two different distributions on the parts of the body **sufficiently far removed** from the region of application of the forces are **essentially the same**, provided that the **two distribution of forces have the same resultant force and the same resultant couple**.*

5.1. The Airy's Stress Function

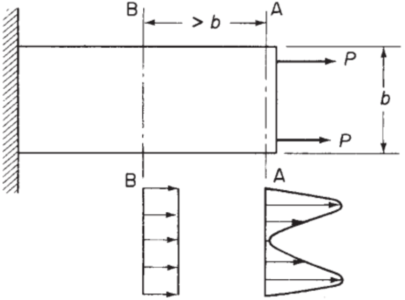
5. 2D Problems

- It may be the ca… the boundary conditions. In th… **ately satisfies the BCs in some s…**
- So is this comple…

St. Venant'… Krempl 2010)

*If **some distribution of** … **y** is replaced by a **different distribution** o… en the **effects** of the two different distributions … rom the region of application of the forc… **distribution of forces have the same resultan…***

Figure from Megson 2013



References I

[1] W. Michael Lai, David Rubin, and Erhard Krempl. Introduction to Continuum Mechanics, 4th ed. Amsterdam Boston: Butterworth-Heinemann/Elsevier, 2010. ISBN: 978-0-7506-8560-3 (cit. on pp. 2, 36, 56, 57).

[2] Martin H. Sadd. Elasticity: Theory, Applications, and Numerics, 2nd ed. Amsterdam ; Boston: Elsevier/AP, 2009. ISBN: 978-0-12-374446-3 (cit. on p. 2).

[3] T. H. G. Megson. Aircraft Structures for Engineering Students, Elsevier, 2013. ISBN: 978-0-08-096905-3 (cit. on pp. 2, 34, 56, 57).