AS3020: Aerospace Structures

Module 3: Introduction to Elasticity

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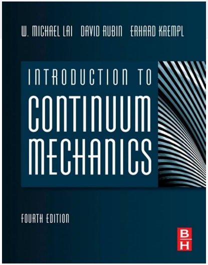
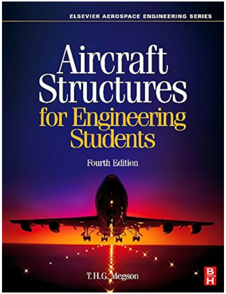
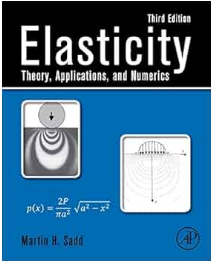


Chapters 1-5 in Lai, Rubin, and Krempl (2010)



Chapters 1,2 in Megson (2013)

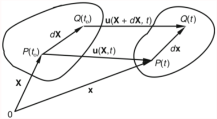


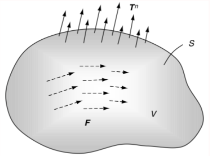
Chapters 1-5 in Sadd (2009)

We have to make a choice!

Red Pill

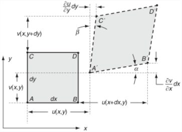
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

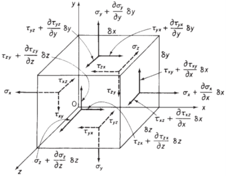




Blue Pill

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}(\sigma_y + \sigma_y)$$





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Red Pill

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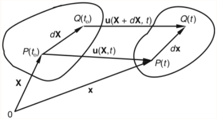

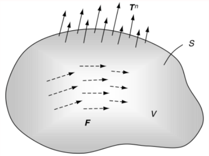

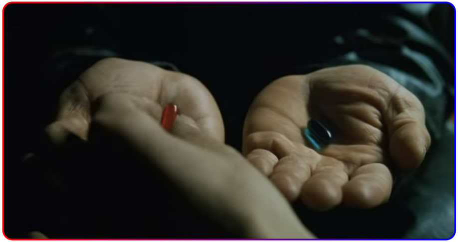

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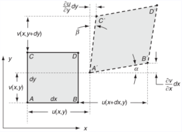

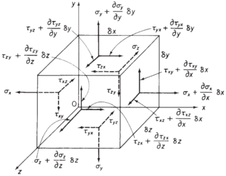

1.1. Vector Convention, Indicial Notation I

1. Mathematical Rudiments

Vector Notation: $\underline{v} = \underset{\sim}{v}^T \underset{\sim}{\underline{e}}$

- We will put a $\underline{(\cdot)}$ underneath a symbol to denote that it is **a vector** (e.g., $\underline{v}$).
- We will put a $\underset{\sim}{(\cdot)}$ underneath a symbol to denote that it is **an array**, i.e., a collection of numbers (e.g., $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$).

  Note that $\underset{\sim}{\underline{e}} = \begin{bmatrix} \underline{e_1} \\ \underline{e_2} \\ \underline{e_3} \end{bmatrix}$, i.e., a collection of unit vectors.
- For tensors we will put two bars below: $\underline{\underline{(\cdot)}}$. Correspondingly, matrices will be written with two tilde underneath $\underset{\approx}{(\cdot)}$.

Einstein's Summation Convention: Dummy Indices

$$s = a_1 x_1 + a_2 x_2 + \cdots = \sum_{i=1}^{n} a_i x_i \rightarrow a_i x_i = a_k x_k = a_m x_m$$

Consider $\alpha = a_{ij} x_i x_j$, $\underline{v} = v_i \underline{e_i}$, $\underline{\underline{T}} = T_{ij} \underline{e_i}\, \underline{e_j}$

1.1. Vector Convention, Indicial Notation II

1. Mathematical Rudiments

Inner Products

- We will use both $\langle \underline{u}, \underline{v} \rangle$ and $\underline{u} \cdot \underline{v}$ to denote the inner product of $\underline{u}$ and $\underline{v}$.
- For tensors we use $\langle \underline{\underline{T}}, \underline{\underline{Q}} \rangle$ and $\underline{\underline{T}} : \underline{\underline{Q}}$ to denote tensor inner products.
- For tensors operating on vectors, we have both the left contraction and right contraction as $\langle \underline{u}, \underline{\underline{T}} \rangle$ and $\langle \underline{\underline{T}}, \underline{u} \rangle$ respectively (also $\underline{u} \cdot \underline{\underline{T}}$ and $\underline{\underline{T}} \cdot \underline{u}$).

Note: $\langle \underline{v}, \underline{u} \rangle = \underline{v} \cdot \underline{u} = \utilde{v}^T \utilde{u}$; $\langle \underline{\underline{T}}, \underline{v} \rangle = \underline{\underline{T}} \cdot \underline{v} = (\utilde{\utilde{T}} \utilde{v})^T \utilde{e}$; $\langle \underline{u}, \langle \underline{\underline{T}}, \underline{v} \rangle \rangle = \underline{u} \cdot \underline{\underline{T}} \cdot \underline{v} = \utilde{u}^T \utilde{\utilde{T}} \utilde{v}$.

Free Indices

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}\right\} \implies y_i = a_{ij}x_j$$

Consider $T_{ij} = A_{im}A_{jm}$.

1.1. Vector Convention, Indicial Notation III

1. Mathematical Rudiments

The Kronecker Delta

$$\delta_{ij} := \langle \underline{e_i}, \underline{e_j} \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $C_{ijkl} = \delta_{ik}\delta_{jl}$,
$C_{ijkl} = \delta_{il}\delta_{jk}$.

The Levi-Civita Symbol

$$\epsilon_{ijk} := \langle \underline{e_i} \times \underline{e_j}, \underline{e_k} \rangle = \begin{cases} 1 & \text{if } \{(i,j,k)\} \in \{(1,2,3),(2,3,1),(3,1,2 \\ -1 & \text{if } \{(i,j,k)\} \in \{(3,2,1),(2,1,3),(1,3,2 \\ 0 & \text{otherwise} \end{cases}$$

Note: $\underline{e_i} \times \underline{e_j} = \epsilon_{ijk}\underline{e_k}$.
Consider $\underline{a} \cdot (\underline{b} \times \underline{c})$, $\Delta\underline{\underline{F}}$.

Property: $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$$\epsilon_{ijk}\epsilon_{mnk} = (\epsilon_{ijk}\underline{e_k}) \cdot (\epsilon_{mnk}\underline{e_k}) = (\underline{e_i} \times \underline{e_j}) \cdot (\underline{e_m} \times \underline{e_n})$$

$$(\underline{e_i} \times \underline{e_j}) \cdot (\underline{e_m} \times \underline{e_n}) = \begin{cases} 1, & \underline{e_i} \times \underline{e_j} = \underline{e_m} \times \underline{e_n} \\ -1, & \underline{e_i} \times \underline{e_j} = -\underline{e_m} \times \underline{e_n} = \underline{e_n} \times \underline{e_m} \\ 0, & \text{otherwise} \end{cases}$$

$$= \boxed{\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}}$$

Consider $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d})$ (Lagrange's identity).

1.1. Vector Convention, Indicial Notation IV

Derivative Notation

$$\underline{\nabla}\,\underline{u} \equiv \frac{\partial u_i}{\partial x_j} := u_{i,j}.$$

In **Operator Notation**, we may write $\underline{\nabla}(\cdot) = \frac{\partial(\cdot)}{\partial x_1}\underline{e_1} + \frac{\partial(\cdot)}{\partial x_2}\underline{e_2} + \frac{\partial(\cdot)}{\partial x_3}\underline{e_3}$.

Exercise: Express the following in indicial notation: $\underline{\nabla}\,\underline{u}$, $\underline{\nabla}\cdot\underline{u}$, $\underline{\nabla}\times\underline{u}$, $\underline{\nabla}\times\underline{\underline{Q}}$, $\underline{\nabla}\,\underline{u}$, $\Delta\underline{u} = \nabla^2\underline{u} = \underline{\nabla}\cdot(\underline{\nabla}\,\underline{u})$, $\underline{\nabla}\cdot(\underline{\nabla}\times\underline{u})$, $\underline{\nabla}\times\underline{\nabla}\times\underline{u}$, $\underline{\nabla}\cdot\underline{\underline{\sigma}}$.

1.2. Matrix Algebra in Indicial Notation

Mathematical Rudiments

Indicial notation leads to some very nifty tricks while dealing with classical matrix algebra. Consider the following:

Determinant of a Matrix is Written as a scalar triple product of its columns or row vectors:

$$\underset{\sim}{\underset{\sim}{A}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \rightarrow A_{ij}.$$

$$det(\underset{\sim}{\underset{\sim}{A}}) = \langle A_{1i}\underline{e_i} \times A_{2j}\underline{e_j}, A_{3k}\underline{e_k}\rangle = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

$$= \langle A_{i1}\underline{e_i} \times A_{j2}\underline{e_j}, A_{k3}\underline{e_k}\rangle = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

Rows(Columns) of the adjoint of a Matrix can be written as the components of the cross product of the remaining Column(Row) vectors

$$Adj(\underset{\sim}{\underset{\sim}{A}})_{1i} = \epsilon_{ijk}A_{j2}A_{k3}, \text{ and}$$

$$Adj(\underset{\sim}{\underset{\sim}{A}})_{i1} = \epsilon_{ijk}A_{2j}A_{3k}.$$

You should be able to verify easily that $Adj(\underset{\sim}{\underset{\sim}{A}})\underset{\sim}{\underset{\sim}{A}} = det(\underset{\sim}{\underset{\sim}{A}})\underset{\sim}{\underset{\sim}{I}}$.

The derivative of the determinant is simplified as

$$\frac{d}{dp}(det(\underset{\sim}{\underset{\sim}{A}})) = \frac{d}{dp}(\epsilon_{ijk}A_{1i}A_{2j}A_{3k}) = \epsilon_{ijk}\left(A'_{1i}A_{2j}A_{3k} + A_{1i}A'_{2j}A_{3k} + A_{1i}A_{2j}A'_{3k}\right)$$

$$= Adj(\underset{\sim}{\underset{\sim}{A}})_{i1}A'_{1i} + Adj(\underset{\sim}{\underset{\sim}{A}})_{j2}A'_{2j} + Adj(\underset{\sim}{\underset{\sim}{A}})_{k3}A'_{3k} = Adj(\underset{\sim}{\underset{\sim}{A}})_{ij}A'_{ji}$$

$$= trace(Adj(\underset{\sim}{\underset{\sim}{A}})_{ij}A'_{jk}) = trace(Adj(\underset{\sim}{\underset{\sim}{A}})\underset{\sim}{\underset{\sim}{A}}').$$

This will turn out to be quite an important result later on.

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Differential Calculus

- Scalar, vector fields
- Gradients, directional derivative
- Divergence, Curl

Observe:

$$\underline{\nabla} \times (\underline{\nabla} \cdot \phi) = \epsilon_{ijk}\phi_{,kj}\underline{e_i}.$$

$$\text{But } \epsilon_{ijk}\phi_{,kj} = -\epsilon_{ikj}\phi_{,jk},$$

So for **continuously differentiable** ϕ, we have

$$\boxed{\epsilon_{ijk}\phi_{,jk} = 0 => \underline{\nabla} \times (\underline{\nabla} \cdot \phi) = 0}.$$

1.3. Some Multi-Variate Calculus

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Curvilinear Coordinates

- Scalar field ϕ gradient:

$$\begin{aligned}\delta\phi &= \frac{\partial\phi}{\partial x_1}\delta x_1 + \frac{\partial\phi}{\partial x_2}\delta x_2\\ &= \frac{\partial\phi}{\partial r}\delta r + \frac{\partial\phi}{\partial\theta}\delta\theta\end{aligned}$$

- Polar bases

$$\underline{e}_r = C_\theta \underline{e}_1 + S_\theta \underline{e}_2 \implies \delta\underline{e}_r = \delta\theta\underline{e}_\theta$$
$$\underline{e}_\theta = -S_\theta \underline{e}_1 + C_\theta \underline{e}_2 \implies \delta\underline{e}_\theta = -\delta\theta\underline{e}_r$$

- Position vector

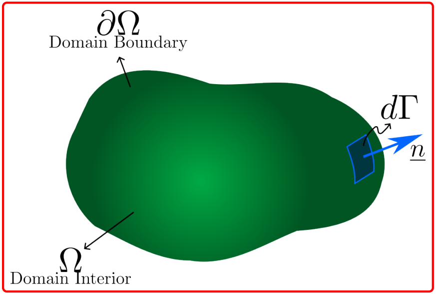
$$\begin{aligned}\delta\underline{r} &= \delta r\underline{e}_r + r\delta\underline{e}_r\\ &= \delta r\underline{e}_r + r\delta\theta\underline{e}_\theta\end{aligned}$$

- For $\delta\phi = \nabla\phi \cdot \delta\underline{r}$,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta$$

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments



Integral Calculus

- **Gauss Divergence Theorem**

$$\int_\Omega F_{,i} d\Omega = \int_{\partial\Omega} \overbrace{\langle F \underline{e_i}, n_j \underline{e_j} \rangle}^{F n_i} d\Gamma$$

This is a general result that works for all objects!

  - Vectors: $\int_\Omega F_{i,j} d\Omega = \int_{\partial\Omega} F_i n_j d\Gamma$. Also $\int_\Omega F_{i,i} d\Omega = \int_{\partial\Omega} F_i n_i d\Gamma$.
  - Tensors: $\int_\Omega F_{ij,k} d\Omega = \int_{\partial\Omega} F_{ij} n_k d\Gamma$

- **Stoke's Law**:

$$\int_{\mathcal{A}} \langle (\underline{\nabla} \times \underline{F}), \underline{n} \rangle d\mathcal{A} = \int_{\partial\mathcal{A}} \langle \underline{F}, \underline{t} \rangle d\ell$$

1.3. Some Multi-Variate Calculus

1. Mathematical Rudiments

Stoke's Law as a Special Case of Gauss Divergence in 2D

$$\int_{\mathcal{A}} \langle \underline{\nabla} \times \underline{F}, \underline{n} \rangle d\mathcal{A} = \int_{\mathcal{A}} \epsilon_{ijk} F_{k,j} n_i d\mathcal{A}$$
$$= \int_{\partial\mathcal{A}} F_k \epsilon_{ijk} n_i b_j d\ell$$
$$= \int_{\partial\mathcal{A}} F_k t_k d\ell$$
$$= \int_{\partial\mathcal{A}} \langle \underline{F}, \underline{t} \rangle d\ell.$$

lus

eorem

$\overbrace{\underline{e_i}, n_j \underline{e_j} \rangle}^{F n_i} d\Gamma$

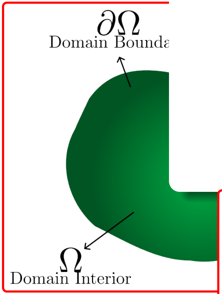
that works for

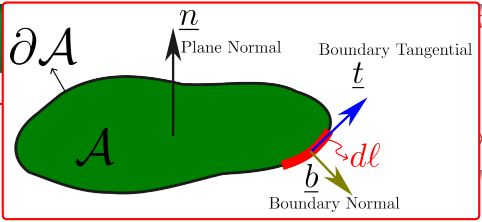
$= \int_{\partial\Omega} F_i n_j d\Gamma.$

$\Omega = \int_{\partial\Omega} F_i n_i d\Gamma.$

$\int_{\partial\Omega} F_{ij} n_k d\Gamma$

$\rangle d\mathcal{A} = \int_{\partial\mathcal{A}} \langle \underline{F}, \underline{t} \rangle d\ell$

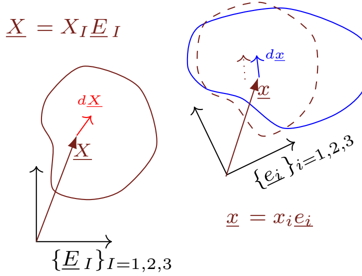




2. Deformations and Strain

2.1. The Basic Premise

How to describe the change in shape **independently** of rigid body motions?



How does $d\underline{X}$ transform into $d\underline{x}$?

$$\underline{x} = \underline{X} + \underline{u}$$

- The deformations are mapped as

$$\begin{aligned} \text{Lagrangian} \quad & x_i = x_i(\underline{X}) \\ \text{Eulerian} \quad & X_i = X_i(\underline{x}) \end{aligned}$$

- Under the **Lagrangian description** we have,

$$dx_i = \overbrace{\frac{\partial x_i}{\partial X_I}}^{F_{iI}} dX_I$$

$$\begin{aligned} \text{Length} \quad & ds^2 = ||d\underline{x}||^2 = dx_i dx_i = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I}\frac{\partial x_i}{\partial X_J}\right] dX_J \\ \text{Angle} \quad & ds_1 ds_2 \cos\theta = dx_i dx_j = \\ & dX_I \left[\frac{\partial x_i}{\partial X_I}\frac{\partial x_j}{\partial X_J}\right] dX_J \end{aligned}$$

2.2. Objectivity and Coordinate Transformation

2. Deformations and Strain

- A vector $\underline{v}$ is written as
$$\underline{v} = v_i \underline{e_i},$$
and is defined as **a linear combination of the bases of its vector-space**.
- Suppose I have another coordinate system spanning the same vector-space, this comes with its own set of basis vectors $\{\underline{e_i}'\}_{i=1,\dots,n}$.
- If the vector represents a physical/geometrical measurement, it **can not change based on coordinate system**, i.e., it is objective.
- So, the following equality must hold:
$$\underline{v} = v_i \underline{e_i} = v_i' \underline{e_i}',$$
with v_i and v_i' being the **components of the same vector** under the different coordinate frames.

2.2. Objectivity and Coordinate Transformation

2. Deformations and Strain

- Assuming that both $\{\underline{e_i}\}$ and $\{\underline{e_i}'\}$ represent **orthogonal rectilinear coordinate systems** (inner products $\langle \underline{e_i}, \underline{e_j} \rangle \equiv \langle \underline{e_i}', \underline{e_j}' \rangle = \delta_{ij}$), we write down:

$$v_i = \langle \underline{e_i}, \underline{v} \rangle; \quad v_i' = \langle \underline{e_i}', \underline{v} \rangle.$$

- Evaluating v_i' we obtain,

$$v_i' = \langle \underline{e_i}', v_j \underline{e_j} \rangle = \langle \underline{e_i}', \underline{e_j} \rangle v_j.$$

  Denoting $\langle \underline{e_i}', \underline{e_j} \rangle = Q_{ij}$, we get our **component tranformation law for a vector**:

$$\boxed{v_i' = Q_{ij} v_j} \Leftrightarrow \boxed{\underset{\sim}{v}' = \underset{\approx}{Q}\underset{\sim}{v}}.$$

- Using the array notation we have $\underline{v} = \underset{\sim}{v}^T \underset{\sim}{e} = \underset{\sim}{v}'^T \underset{\sim}{e}'$. Substituting the above we can show that the basis vectors themselves also transform (assuming rectilinear transformations) as

$$\boxed{\underset{\sim}{e}' = \underset{\approx}{Q}\underset{\sim}{e}}.$$

2.2. Objectivity and Coordinate Transformation: Tensors

2. Deformations and Strain

- We will define a (2nd order) tensor as a **linear combination of basis-dyads**:

$$\underline{\underline{T}} = T_{ij}\underline{e_i}\,\underline{e_j} = T'_{ij}\underline{e_i}'\underline{e_j}',$$

  where **we have required $\underline{\underline{T}}$ to be invariant** under coordinate change.

- Using a **double-contraction** operation (dyadic inner product), we write down the components of T'_{ij} as,

$$\begin{aligned} T'_{ij} &= \langle \underline{\underline{T}}, \underline{e_i}'\underline{e_j}' \rangle = T_{mn}\langle \underline{\underline{e_m\,e_n}}, \underline{e_i}'\underline{e_j}' \rangle \\ &= T_{mn} \underbrace{\langle \underline{e_i}', \underline{e_m} \rangle}_{Q_{im}} \overbrace{\langle \underline{e_j}', \underline{e_n} \rangle}^{Q^{jn}} \\ &= Q_{im}T_{mn}Q_{jn}. \end{aligned}$$

- In array notation we write the components as,

$$\boxed{\underset{\sim}{T}' = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T}.$$

For a tensor to be invariant, its components have to transform in this fashion.

2.2. Objectivity and Coordinate Transformation: Summary

2. Deformations and Strain

Supposing I specify a basis change by

$$\underset{\sim}{e}' = \underset{\approx}{Q}\underset{\sim}{e},$$

- for a vector $\underline{v} = \underset{\sim}{v}^T \underset{\sim}{e}$ to be invariant, its components have to transform as

$$\underset{\sim}{v}' = \underset{\approx}{Q}\underset{\sim}{v}.$$

- for a tensor $\underline{\underline{T}} = \underset{\approx}{T}\underset{\sim}{e} \otimes \underset{\sim}{e}$ to be invariant, its components have to transform as

$$\underset{\approx}{T}' = \underset{\approx}{Q}\underset{\approx}{T}\underset{\approx}{Q}^T$$

- If it transforms in any other fashion, **then invariance is not guaranteed**, or in other words, **the quantity is not objective**.

2.2. Objectivity and Coordinate Transformation: Relationship to Gradients

2. Deformations and Strain

We will now establish a relationship between coordinate transformation and **component-gradients**.

- Consider an infinitesimal line vector $d\underline{x} = dx_i \underline{e_i} = dx'_i \underline{e_i}'$.
- It is obvious that the components $d\underset{\sim}{x}'$ have to be related to the components $d\underset{\sim}{x}$. So we write

$$dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j \tag{1}$$

- By invariance requirements, we have

$$dx'_i = Q_{ij} dx_j. \tag{2}$$

- Comparing eq. (1) and eq. (2) we obtain,

$$\boxed{Q_{ij} = \frac{\partial x'_i}{\partial x_j}} \quad \text{or} \quad \boxed{\underset{\approx}{Q} = grad\left(\underset{\sim}{x}'\right)}$$

$grad(\cdot)$ operator $\Longrightarrow$ gradient operation

2.2. Objectivity and Coordinate Transformation: The Deformation Gradient

2. Deformations and Strain

- The deformation gradient ($F_{iI} = \frac{\partial x_i}{\partial X_I}$) relates quantities in the deformed (x_i) and the undeformed configurations (X_I). So we shall investigate the influence of coordinate change on it.
- We setup coordinate change as $\underset{\approx}{\underline{E}} \to \overline{\underset{\approx}{\underline{E}}}$ (undeformed configuration coordinate change) and $\underset{\approx}{\underline{e}} \to \underset{\approx}{\underline{e}}'$ (deformed configuration coordinate change) such that the coordinate transformation matrices are

$$\boxed{Q_{IJ}^{(X)} = \frac{\partial \overline{X_I}}{\partial X_J} = \langle \overline{\underline{E_I}}, \underline{E_J} \rangle} \text{ and } \boxed{Q_{ij}^{(x)} = \frac{\partial x_i'}{\partial x_j} = \langle \underline{e_i}', \underline{e_j} \rangle}.$$

- Under this coordinate change we have,

We assume orthonormal rectilinear bases, so $\underset{\approx}{Q}^{-1} = \underset{\approx}{Q}^T$.

$$\overline{F}'_{iI} = \frac{\partial x_i'}{\partial \overline{X_I}} = \frac{\partial x_i'}{\partial x_j} \frac{\partial x_j}{\partial X_J} \frac{\partial X_J}{\partial \overline{X_I}}$$

$$= Q_{ij}^{(x)} F_{jJ} (\underset{\approx}{Q}^{(X)^T})_{JI} \implies \boxed{\overline{\underset{\approx}{F}}' = \underset{\approx}{Q}^{(x)} \underset{\approx}{F} \underset{\approx}{Q}^{(X)^T}}.$$

This is transforming quite unlike a tensor

$\underset{\approx}{Q}^{(x)}$ and $\underset{\approx}{Q}^{(X)}$ **need not necessarily be the same** (we are free to choose measurement coordinates at each instant)

2.2. Objectivity and Coordinate Transformation: The Cauchy Deformation Tensor

2. Deformations and Strain

- Now we consider $\underset{\approx}{C} = \underset{\approx}{F}^T \underset{\approx}{F}$. Under coordinate change this becomes,

$$\overline{\underset{\approx}{C}} = \overline{\underset{\approx}{F}}'^T \overline{\underset{\approx}{F}}' = \left(\underset{\approx}{Q}^{(x)} \underset{\approx}{F} \underset{\approx}{Q}^{(X)^T}\right)^T \left(\underset{\approx}{Q}^{(x)} \underset{\approx}{F} \underset{\approx}{Q}^{(X)^T}\right)$$

$$= \underset{\approx}{Q}^{(X)} \underset{\approx}{F}^T \underset{\approx}{Q}^{(x)^T} \underset{\approx}{Q}^{(x)} \underset{\approx}{F} \underset{\approx}{Q}^{(X)^T}$$

$$\implies \boxed{\underset{\approx}{C} = \underset{\approx}{Q}^{(X)} \underset{\approx}{F}^T \underset{\approx}{F} \underset{\approx}{Q}^{(X)^T}}$$

Unlike the deformation gradient...

...this is transforming like a tensor's components!
So it would make sense to define a tensor of the form $\underline{\underline{C}} = C_{IJ} \underline{E}_I \underline{E}_J$. This is referred to as the **Cauchy deformation tensor**.

Note that all of the above is merely an aside, motivating the construction of an "objective representation" of deformation. We will next see how this is practically useful.

2.3. The Strain Tensor

2. Deformations and Strain

- We are now ready to define the strain tensor based on length change. Denoting $||d\underline{X}|| = dS$ and $||d\underline{x}|| = ds$ we write,

$$ds^2 - dS^2 = dX_I \left(F_{iI}F_{jJ} - \delta_{IJ}\right) dX_J = d\underset{\sim}{X}^T \left[\underset{\approx}{F}^T \underset{\approx}{F} - \underset{\approx}{I}\right] d\underset{\sim}{X}$$

You should be comfortable with the notation in this equality by now!

$$= d\underset{\sim}{X}^T \left[\underset{\approx}{C} - \underset{\approx}{I}\right] d\underset{\sim}{X}$$

$$= d\underline{X} \cdot \left(\underline{\underline{C}} - \underline{\underline{I}}\right) \cdot d\underline{X}.$$

- For small changes in length, $ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$. So the above equation becomes

$$2dS(ds - dS) = d\underline{X} \cdot \left[\underline{\underline{C}} - \underline{\underline{I}}\right] \cdot d\underline{X} \implies \boxed{\frac{ds - dS}{dS} = \left(\frac{d\underline{X}}{dS}\right) \cdot \left(\frac{1}{2}\left[\underline{\underline{C}} - \underline{\underline{I}}\right]\right) \cdot \left(\frac{d\underline{X}}{dS}\right)}.$$

- We hereby come across a convenient **objective tensor quantity**:

$$\boxed{\underline{\underline{\mathcal{E}}} = \frac{1}{2}\left[\underline{\underline{C}} - \underline{\underline{I}}\right]}.$$

Formally $\underline{\underline{\mathcal{E}}}$ is known as the Green Lagrange Strain Tensor.

2.3. The Strain Tensor: Interpretations

Deformations and Strain

- Let us consider $\underline{dX} = dS\underline{E_1}$, i.e., in the undeformed configuration the line segment is along $\underline{e_1}$. The relative length change for this can be written as

$$\frac{ds - dS}{dS} = (\underline{E_1}) \cdot \underline{\underline{\mathcal{E}}} \cdot (\underline{E_1}) = \mathcal{E}_{11}.$$

I.e., $\mathcal{E}_{11}$ represents the relative elongation of a line segment along the $\underline{E_1}$ direction in the undeformed state.
(Similarly $\mathcal{E}_{22}, \mathcal{E}_{33}$ can be interpreted)

- So the diagonal elements of $\underline{\underline{\mathcal{E}}}$ represent relative elongations, a.k.a., "straight strains".

How about **shape change**?

2.3. The Strain Tensor: Interpretations

2. Deformations and Strain

- For considering shape changes, it is **not enough just to look at a single line-segment**.
- Let us consider 2 line-vectors along the $\underline{E_1}$ and $\underline{E_2}$ vectors: $d\underline{X}^{(A)} = dS^{(A)}\underline{E_1}$, $d\underline{X}^{(B)} = dS^{(B)}\underline{E_2}$ (so we have $d\underline{X}^{(A)} \cdot d\underline{X}^{(B)} = 0$).
- After deformation, the angle between the two, θ can be obtained through the inner product:

$$d\underline{x}^{(A)} \cdot d\underline{x}^{(B)} = ds^{(A)} ds^{(B)} \cos\theta.$$

- Writing $\theta = \frac{\pi}{2} - \gamma_{12}$ and $ds^{(A)} = (1+\mathcal{E}_{11})dS^{(A)}$ (similarly for $ds^{(B)}$), the above simplifies as

$$d\underline{X}^{(A)} \cdot \underline{\underline{C}} \cdot d\underline{X}^{(B)} = dS^{(A)} dS^{(B)} (1+\mathcal{E}_{11})(1+\mathcal{E}_{22}) \sin\gamma_{12}$$

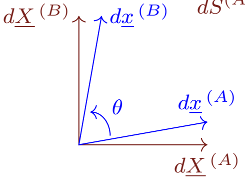
$\sin\gamma_{12} \approx \gamma_{12}$

Small Deformation

$$\begin{bmatrix} dS^{(A)} & 0 & 0 \end{bmatrix} \begin{bmatrix} 2\underset{\approx}{\mathcal{E}} + \underset{\approx}{I} \end{bmatrix} \begin{bmatrix} 0 \\ dS^{(B)} \\ 0 \end{bmatrix} \approx dS^{(A)} dS^{(B)} (\gamma_{12} + \cancel{\mathcal{E}_{11}\gamma_{12} + \mathcal{E}_{22}\gamma_{12} + \mathcal{E}_{11}\mathcal{E}_{22}\gamma_{12}})$$

$$dS^{(A)} dS^{(B)} 2\mathcal{E}_{12} = dS^{(A)} dS^{(B)} \gamma_{12}.$$

$$\boxed{\gamma_{12} = 2\mathcal{E}_{12}}.$$

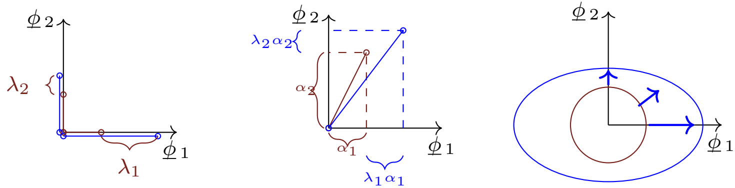


We call this the **shear strain**.

2.3. The Strain Tensor: Interpretations

2. Deformations and Strain

- Consider the operation $\underset{\approx}{\mathcal{E}}\underset{\sim}{u}$. Say, $\underset{\sim}{v} = \underset{\approx}{\mathcal{E}}\underset{\sim}{u}$.
  $\underset{\sim}{v}$ represents the **components of a vector** which can be arbitrarily oriented w.r.t. $\underset{\sim}{u}$.
- Consider some unit vector $\underset{\sim}{\phi}$ such that $\underset{\approx}{\mathcal{E}}\underset{\sim}{\phi} = \lambda\underset{\sim}{\phi}$.
  The operation of the matrix $\underset{\approx}{\mathcal{E}}$ leads to perfect stretching by a factor of λ.
- The pair $(\lambda, \underset{\sim}{\phi})$ are known as an **eigenpair** of $\underset{\approx}{\mathcal{E}}$ and $\underset{\sim}{\phi}$ represents a principal direction.
- For 3D mechanics, we have 3 principal directions.
  Consider the 2D case below:



2.3. The Strain Tensor: In terms of displacement

2. Deformations and Strain

Let us now express strain in terms of the displacement field $\underline{u}(\underline{X})$.

- We have $x_i = X_i + u_i$. So the deformation gradient is written as,

$$F_{iI} = \frac{\partial x_i}{\partial X_I} = \delta_{iI} + u_{i,I} \Leftrightarrow \underset{\approx}{F} = \underset{\approx}{I} + \underset{\approx}{\nabla u}.$$

- Cauchy deformation tensor is written as (with components $\underset{\approx}{C} = \underset{\approx}{F}^T \underset{\approx}{F}$),

$$C_{IJ} = F_{iI}F_{iJ} = \delta_{IJ} + u_{I,J} + u_{J,I} + u_{i,I}u_{i,J}.$$

- From this, the strain tensor is written as (with components $\underline{\underline{\mathcal{E}}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$)

$$\mathcal{E}_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \underbrace{\frac{\partial u_i}{\partial X_I}\frac{\partial u_i}{\partial X_J}}_{\text{ignored for small strain}}\right)$$

- **Infinitesimal Strain Tensor:** $\boxed{\mathcal{E}_{IJ} = \frac{1}{2}(u_{I,J} + u_{J,I})}$.

2.3. The Strain Tensor: Volume Change

2. Deformations and Strain

- Consider three arbitrarily oriented vectors $d\underset{\sim}{X}^{(1)}$, $d\underset{\sim}{X}^{(2)}$, $d\underset{\sim}{X}^{(3)}$ in the undeformed configuration. The volume that they describe is given by

$$dV = \epsilon_{IJK} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}.$$

- Upon deformation, using the same notation as above, the volume **becomes**

$$dv = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}.$$

Using the deformation gradient to write this out ($d\underset{\sim}{x} = \underset{\approx}{F} d\underset{\sim}{X}$), we have

$$dv = \underbrace{\epsilon_{ijk} F_{iI} F_{jJ} F_{kK}} dX_I^{(1)} dX_J^{(2)} dX_K^{(3)}$$

- We have previously seen that $\epsilon_{ijk} F_{iI} F_{jJ} F_{kK} = \epsilon_{IJK} det(\underset{\approx}{F})$. Substituting this in the above we get,

$$dv = \epsilon_{IJK} det(\underset{\approx}{F}) dX_I^{(1)} dX_J^{(2)} dX_K^{(3)} = det(\underset{\approx}{F}) dV.$$

- $J := det(\underset{\approx}{F})$ is known as the *Jacobi determinant*. $\boxed{dv = JdV}$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- For the infinitesimal case, the deformation gradient component matrix is expressed as

$$\underset{\sim}{\underset{\sim}{F}} = \underset{\sim}{\underset{\sim}{I}} + \varepsilon \underset{\sim}{\underset{\sim}{\nabla u}},$$

where $\varepsilon > 0$ is **some small number** ($\varepsilon \ll 1$).

- Since ε is small, we will try to expand out J as a Taylor series in ε about $\varepsilon = 0$:

$$J(\varepsilon) = J(\varepsilon = 0) + \varepsilon \left.\frac{dJ}{d\epsilon}\right|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2).$$

Derivative of Determinant

$$\frac{d}{dp}\left(det(\underset{\sim}{\underset{\sim}{M}})\right) = trace\left(Adj(\underset{\sim}{\underset{\sim}{M}})\frac{d\underset{\sim}{\underset{\sim}{M}}}{dp}\right)$$

For invertible $\underset{\sim}{\underset{\sim}{M}}$, $Adj(\underset{\sim}{\underset{\sim}{M}}) = J\underset{\sim}{\underset{\sim}{M}}^{-1}$.

- This simplifies as,

$$J(\varepsilon) = det(\underset{\sim}{\underset{\sim}{I}}) + \varepsilon\left(J(\varepsilon = 0)trace\left(\underset{\sim}{\underset{\sim}{I}}^{-1}\underset{\sim}{\underset{\sim}{\nabla u}}\right)\right) + \mathcal{O}(\varepsilon^2) \approx 1 + \varepsilon \overbrace{tr(\underset{\sim}{\underset{\sim}{\nabla u}})}^{\underline{\nabla} \cdot \underline{u} = tr(\underset{\sim}{\underset{\sim}{\mathcal{E}}})}$$

2.3. The Strain Tensor: Infinitesimal Volume Change

2. Deformations and Strain

- Undeformed volume is $\boxed{dV}$, deformed volume is $\boxed{dv = JdV}$. So **relative change in volume is**
$$\frac{dv - dV}{dV} = J - 1.$$
- For the infinitesmial displacement case $J \approx 1 + tr(\nabla u)$ (we have set $u \to \varepsilon u$ here). Substituting, we get
$$\frac{dv - dV}{dV} = tr(\nabla u) = u_{I,I} = \mathcal{E}_{II} = tr(\underset{\sim}{\mathcal{E}}).$$
- So the **trace of the strain tensor** is the relative volume change.

In Summary we have, for the strain tensor,

- Each diagonal element corresponds to **stretching/compressing**,
- Off-diagonal elements correspond to **shearing**,
- Trace (sum of diagonal elements) corresponds to **volume change**.

Summary

2. Deformations and Strain

- We have defined the deformation gradient **matrix** $\underset{\sim}{\underset{\sim}{F}}$ and the strain **tensor** $\underline{\underline{\mathcal{E}}}$.
- **Notice:** Under no deformation, if you just changed the coordinate **frame of observation**, $\underset{\sim}{\underset{\sim}{F}}$ will change, **but** $\underset{\sim}{\underset{\sim}{\mathcal{E}}}$ **will not**.

Rigid Body Motion

$$\underline{x} = \underline{c} + \underline{\underline{R}}\,\underline{X}$$

- What is the deformation gradient here?
- What is the **infinitesimal strain tensor** here?
- What is the **finite strain tensor** here?

- What should the material **respond to**? What is the quantity that the **material wants to resist**?

Additional Reading:
Einstein's Covariance Principle

2.4. Strain Compatibility

2. Deformations and Strain

Necessary Reading

Read Section 1.10 in Megson (2013)

- Since strains are defined **based on the displacement field**, the different strain components are related.
- For the infinitesimal case we have: $2\mathcal{E}_{IJ} = u_{I,J} + u_{J,I}$. To avoid confusion with the Levi-Civita symbol we will use $\underline{\underline{\mathcal{E}}}$ to denote the strain tensor henceforth.
  We want to manipulate this such that we get an **equality fully expressed in the strains alone.**
- Differentiating by X_K and premultiplying by ϵ_{MJK} we have,

$$2\epsilon_{MJK}\mathcal{E}_{IJ,K} = \underbrace{\epsilon_{MJK}u_{I,JK}}_{0} + \epsilon_{MJK}u_{J,IK} \rightarrow \boxed{\text{free indices: } I, M}$$

- We differentiate this by X_L and pre-multiply by ϵ_{NIL} to get:

$$2\epsilon_{NIL}\epsilon_{JKM}\mathcal{E}_{IJ,KL} = \underbrace{\epsilon_{MJK}\epsilon_{NIL}u_{J,IKL}}_{0} \rightarrow \boxed{\text{free indices: } K, L}$$

2.4. Strain Compatibility

2. Deformations and Strain

- The compatibility equation $\boxed{\epsilon_{MJK}\epsilon_{NIL}E_{IJ,KL} = 0}$ represents a 3×3 system of 9 equations.
- We have **two symmetries**: $E_{IJ} = E_{JI}$ (strain tensor symmetry), and $E_{IJ,KL} = E_{IJ,LK}$ (strain continuously differentiable).
  Applying this can convince us that the equation is also symmetric. So we have $\frac{3(3+1)}{2} = 6$ **unique equations**.
- In component notation, these can be written out as,

$$\underbrace{E_{22,33} + E_{33,22} = 2E_{23,23}}_{(M,N)=(1,1)}, \quad \underbrace{E_{22,13} + E_{13,22} = E_{12,23} + E_{23,12}}_{(M,N)=(1,3)}$$

$$\underbrace{E_{33,11} + E_{11,33} = 2E_{13,13}}_{(M,N)=(2,2)}, \quad \underbrace{E_{33,12} + E_{12,33} = E_{13,23} + E_{23,13}}_{(M,N)=(1,2)}$$

$$\underbrace{E_{11,22} + E_{22,11} = 2E_{12,12}}_{(M,N)=(3,3)}, \quad \underbrace{E_{11,23} + E_{23,11} = E_{12,13} + E_{13,12}}_{(M,N)=(2,3)}$$

The strains have to satisfy these conditions for them to "have been generated" by a continuously differentiable displacement field.

2.5. Illustrative Example

Deformations and Strain

Also try to get expressions for how areas and volumes will transform!

- Displacement Field ($\varepsilon > 0$ some small number):

$$\underline{u} = \varepsilon \begin{bmatrix} 0.1X_1 + 0.8X_2\sin(4X_1) \\ 0.2(\cos(4X_1) - 1) \\ 0 \end{bmatrix}$$

- Deformation Gradient ($\underset{\approx}{F} = \underset{\approx}{I} + \nabla \underline{u}$):

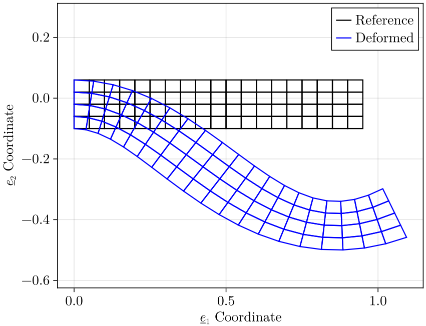
$$\underset{\approx}{F} = \varepsilon \begin{bmatrix} 1.1 + 3.2X_2\cos(4X_1) & 0.8\sin(4X_1) & 0 \\ -0.8\sin(4X_1) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Infinitesimal Strain Tensor components:

$$\underset{\approx}{\mathcal{E}} = \varepsilon \begin{bmatrix} 0.1 + 3.2X_2\cos(4X_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Stretched length of axial "fibre" (initially along $\underline{e_1}$) is

$$\ell_{def} = \int_0^1 (1+\mathcal{E}_{11}) dX_1 = 1 + \varepsilon\,(0.1 + 0.8X_2\sin(4)).$$



Deformed configuration plotted for $\varepsilon = 1$. A discretized wire-mesh is chosen just for plotting.

3. Stress and Equilibrium

Force is a vector. Area is a vector. What is **pressure** (F/A)?

- Consider a small area ΔA in a cut-section of an elastic body as shown. The **traction vector** $\underline{t}$ is the limiting force

$$\underline{t} = \lim_{\Delta A \to 0} \frac{\Delta \underline{F}}{\Delta A}.$$

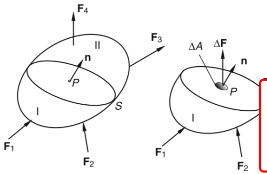


Figure from Lai, Rubin, and Krempl 2010

Cauchy Stress Principle

1. $\underline{t}(-\underline{n}) = -\underline{t}(\underline{n})$.
2. $\underline{t}\left(\sum_{i=1}^{3} \Delta A_i \underline{e_i}\right) = \sum_{i=1}^{3} \underline{t}(\Delta A_1 \underline{e_1})$.

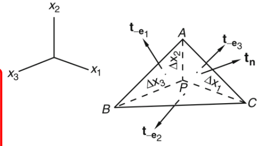


Figure from Lai, Rubin, and Krempl 2010

- By basic force-balance arguments, we can argue that the relationship between the traction vector and the normal vector to the chosen area **has to be linear**.

$$\boxed{t_i = \sigma_{ij} n_j} \Leftrightarrow \boxed{\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}}.$$

Cauchy Stress Tensor: $\underline{\underline{\sigma}} = \sigma_{ij} \underline{e_i}\, \underline{e_j}$ s.t. $\underline{t} = t_i \underline{e_i} = \sigma_{ij} n_j \underline{e_i} = \underline{\underline{\sigma}} \cdot \underline{n}$.

Force Equilibrium

3. Stress and Equilibrium

- Consider the forces on a small volume dv in the **deformed domain** (denoted Ω_d):
  Body loads $\int_{\Omega_d} f_i(\underline{x})dv$
  Surface tractions $\int_{\partial\Omega_d} t_i d|a|$
- Static equilibrium is written as

$$\int_{\partial\Omega_d} \sigma_{ij} da_j + \int_{\Omega_d} f_i dv = 0.$$

- Applying Gauss divergence, this simplifies to,

$$\int_{\Omega_d} \sigma_{ij,j} + f_i dv = 0 \implies \boxed{\sigma_{ij,j} + f_i = 0}.$$

- This is the **static equilibrium equation**.

Moment Equilibrium

3. Stress and Equilibrium

- We next consider the balance of the moments of forces on the same differential element.

$$\int_{\partial\Omega_d} \underbrace{\epsilon_{ijk}x_j\sigma_{kl}da_l}_{\underline{x}\times\underline{t}\,d|a|} + \int_{\Omega_d} \epsilon_{ijk}x_j f_k dv = 0.$$

- Applying Gauss divergence again we get,

$$\int_{\Omega_d} \epsilon_{ijk}((x_j\sigma_{kl})_{,l} + x_j f_k)dv = \int_{\Omega_d} \epsilon_{ijk}\left(\delta_{jl}\sigma_{kl} + x_j\cancel{(\sigma_{kl,l} + f_k)}\right)dv = 0$$

$$\implies \boxed{\epsilon_{ijk}\sigma_{jk} = 0}$$

  which is an assertion of **symmetry of the stress tensor**.
- Note that we have assumed the absense of body moments here.

3.1. Stress Work Done

3. Stress and Equilibrium

- Let us now consider the work done by the stress. For convenience, we start with the **rate of work done**: force×velocity .
- On the infinitesimal element we have,

$$\frac{dU}{dt} = \int_{\partial\Omega_d} \sigma_{ij}\dot{u}_i da_j + \int_{\Omega_d} f_i \dot{u}_i dv.$$

- Application of Gauss divergence leads to,

$$\frac{dU}{dt} = \int_{\Omega_d} \left(\sigma_{ij}\dot{u}_i\right)_{,j} + f_i\dot{u}_i dv = \int_{\Omega_d} \sigma_{ij}\dot{u}_{i,j} + \dot{u}_i \cancel{(\sigma_{ij,j} + f_i)} dv$$

$$\implies \frac{dU}{dt} = \int_{\Omega_d} \sigma_{ij}\frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) dv = \int_{\Omega_d} \sigma_{ij}\mathcal{E}_{ij} dv.$$

- The **power density is written as**,

$$\boxed{\frac{d\mathcal{U}}{dt} = \sigma_{ij}\dot{\mathcal{E}}_{ij}}.$$

3.1. Stress Work Done: Non-Dissipative Solid

3. Stress and Equilibrium

Additional Reading: What is an **Exact Differential**?

- For a general non-dissipative solid, the work done must be path-independent, i.e., the energy U must be a state of the system. In other words, the energy must be solely dependent on the system's configuration, i.e., kinematic state.
- So we shall write

$$\frac{dU}{dt} = \frac{\partial U}{\partial \mathcal{E}_{ij}} \dot{\mathcal{E}}_{ij}.$$

- The above also holds for the energy density $\mathcal{U}$ and we already have $\dot{\mathcal{U}} = \sigma_{ij}\dot{\mathcal{E}}_{ij}$. So

$$\frac{\partial \mathcal{U}}{\mathcal{E}_{ij}} \dot{\mathcal{E}}_{ij} = \sigma_{ij}\dot{\mathcal{E}}_{ij} \implies \boxed{\sigma_{ij} = \frac{\partial \mathcal{U}}{\mathcal{E}_{ij}}}.$$

- In other words, **it MUST be possible to write the stress as a gradient of a scalar energy density with respect to strain for non-dissipativity**.
- So we say that **stress and strain are energy conjugates of one another**.

Technical Note

In the above derivation, we have made some rather sweeping assumptions about the deformations and the deformation gradients being small. For e.g., the deformed domain Ω_d is taken to be approximately the same as the un-deformed domain Ω and the infinitesimal strain formula is directly invoked. See sec. 4.12 in Lai, Rubin, and Krempl 2010 for the general case.

4. Constitutive Relationships

- We have developed tensor-representations of both the stress, $\underline{\underline{\sigma}} = \sigma_{ij}\underline{e_i}\,\underline{e_j}$ and strain, $\underline{\underline{\mathcal{E}}} = \mathcal{E}_{ij}\underline{e_i}\,\underline{e_j}$. **We are now interested in relating the components of the two.**
- The most general **linear** relationship that one can assume is

$$\sigma_{ij} = C_{ijkl}\mathcal{E}_{kl}.$$

- If the system is **non-dissipative**, then the stress must be expressible as $\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial \mathcal{E}_{ij}}$. So,

$$\frac{\partial^2 \mathcal{U}}{\partial \mathcal{E}_{ij}\partial \mathcal{E}_{kl}} = C_{ijkl}.$$

- Since we expect a smooth energy density, the indices (i, j) and (k, l) must be swappable. This represents the first symmetry property of C_{ijkl} $(i, j \leftrightarrow k, l)$.
- Since stress and strain are also symmetric, the following index-swaps must be permissible: $i \leftrightarrow j$, $k \leftrightarrow l$.

Simplification Arguments

4. Constitutive Relationships

- In summary we have the following roadmap for simplification:

General Case	C_{ijkl}	$3 \times 3 \times 3 \times 3 = 81$ terms
Stress-Strain Symmetry	$i \leftrightarrow j,\ k \leftrightarrow l$	$\frac{3(3+1)}{2} \times \frac{3(3+1)}{2} = 36$ terms
Non-dissipativity, smoothness	$(i,j) \leftrightarrow (k,l)$	$\frac{6(6+1)}{2} = 21$ terms

- Suppose the **material is isotropic**, then the components C_{ijkl} are invariant under coordinate transformations. This means that it must be composed of $\delta_{..}$ symbols.
- Under symmetry, we have 3 unique combinations:

$$\delta_{ij}\delta_{kl}, \quad \delta_{ik}\delta_{jl}, \quad \delta_{il}\delta_{jk},$$

  and we write:

$$C_{ijkl} = \alpha_1 \delta_{ij}\delta_{kl} + \alpha_2 \delta_{ik}\delta_{jl} + \alpha_3 \delta_{il}\delta_{jk}.$$

- Applying this to the stress-strain relationship, we get:

$$\sigma_{ij} = \alpha_1 \delta_{ij}\mathcal{E}_{kk} + \alpha_2 \mathcal{E}_{ij} + \alpha_3 \mathcal{E}_{ji} \quad \Longrightarrow \quad \boxed{\sigma_{ij} = \lambda \delta_{ij}\mathcal{E}_{kk} + 2\mu \mathcal{E}_{ij}}.$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Consider a 2D case with $\underset{\sim}{\underset{\sim}{\sigma}} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}$.
- Consider a plane section with normal $\underset{\sim}{\hat{n}} = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}^T$. The perpendicular is denoted $\underset{\sim}{\hat{s}} = \begin{bmatrix}-\sin\theta & \cos\theta\end{bmatrix}^T$.
- The traction vector is given by $\underset{\sim}{t} = \underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{\hat{n}}$:

$$\underset{\sim}{t} = \begin{bmatrix}\sigma_{11}\cos\theta + \sigma_{12}\sin\theta\\ \sigma_{12}\cos\theta + \sigma_{22}\sin\theta\end{bmatrix}.$$

- This is resolved along the $(\underset{\sim}{\hat{n}}, \underset{\sim}{\hat{s}})$ directions by the coordinate transformation,

$$\begin{bmatrix}\sigma_n\\ \tau_s\end{bmatrix} = \begin{bmatrix}\underset{\sim}{\hat{n}}^T\\ \underset{\sim}{\hat{s}}^T\end{bmatrix}\underset{\sim}{t} = \begin{bmatrix}\frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta\\ -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- Now we consider two infinitesimal lines initially oriented along $\underline{n}$ and $\underline{t}$ ($dS\underline{n}$, $dS\underline{t}$).
- $dS\underline{n}$ experiences the elongation,

$$\frac{ds - dS}{dS} = \underline{n} \cdot \underline{\underline{\mathcal{E}}} \cdot \underline{n} = \varepsilon_n.$$

- The shear strain between them is,

$$\gamma_s = 2\underline{t} \cdot \underline{\underline{\mathcal{E}}} \cdot \underline{n}.$$

- Simplifying, we get

$$\begin{bmatrix} \varepsilon_n \\ \gamma_s \end{bmatrix} = \begin{bmatrix} \frac{\mathcal{E}_{11}+\mathcal{E}_{22}}{2} + \frac{\mathcal{E}_{11}-\mathcal{E}_{22}}{2}\cos 2\theta + \mathcal{E}_{12}\sin 2\theta \\ -(\mathcal{E}_{11} - \mathcal{E}_{22})\sin 2\theta + 2\mathcal{E}_{12}\cos 2\theta \end{bmatrix}$$

4.1. Mohr's Circles

4. Constitutive Relationships

- When no shear load/response is observed, these reduce to,

$$\boxed{\begin{bmatrix}\sigma_n\\ \tau_s\end{bmatrix} = \begin{bmatrix}\frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta\\ -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta\end{bmatrix}}, \quad \boxed{\begin{bmatrix}\varepsilon_n\\ \gamma_s\end{bmatrix} = \begin{bmatrix}\frac{\mathcal{E}_{11}+\mathcal{E}_{22}}{2} + \frac{\mathcal{E}_{11}-\mathcal{E}_{22}}{2}\cos 2\theta\\ -(\mathcal{E}_{11}-\mathcal{E}_{22})\sin 2\theta\end{bmatrix}}.$$

- For a linear-elastic material, **causal links** may be made between $\sigma_n \leftrightarrow \varepsilon_n$ and $\tau_s \leftrightarrow \gamma_s$.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- From basic arguments one can motivate

$$\mathcal{E}_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}).$$

- For the 2D case under pure tension,

$$\mathcal{E}_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad \mathcal{E}_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22}.$$

- For some section oriented by angle θ we have,

$$\gamma_s(\theta) = -(\mathcal{E}_{11} - \mathcal{E}_{22})\sin 2\theta = -\frac{1+\nu}{E}\underbrace{(\sigma_{11} - \sigma_{22})\sin 2\theta}_{2\tau_s},$$

which implies, $\boxed{\gamma_s(\theta) = 2\frac{1+\nu}{E}\tau_s}$.

- E: Young's Modulus, ν: Poisson's Ratio, and $G = \frac{E}{2(1+\nu)}$: Shear Modulus.

4.2. Linear Isotropic Elasticity

4. Constitutive Relationships

- We have also spoken about volume change. In terms of strains this is,

$$\frac{dv - dV}{dv} = \mathcal{E}_{11} + \mathcal{E}_{22} + \mathcal{E}_{33}$$
$$= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}).$$

- In other words we have $\mathcal{E}_{ii} = \kappa\sigma_{ii}$, where $\boxed{\kappa = \frac{1-2\nu}{E}}$, the bulk modulus.
- From physical arguments, it is clear that $\kappa > 0$, which implies $\nu < 0.5$, which presents an **upper bound for the Poisson's ratio**.
- The shear modulus must also be positive. So we have $\frac{E}{2(1+\nu)} > 0$, which implies $\boxed{\nu > -1}$, which presents a **lower bound for the Poisson's ratio**.
- In summary we have, $\boxed{\nu \in (-1, 0.5)}$, $\boxed{E > 0}$.

Constitutive Relationship

In summary, the constitutive relationship can be written as,

$$\mathcal{E}_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right].$$

5. 2D Problems

- In 2D, the governing equations can be written as,

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0$$
$$\sigma_{12,1} + \sigma_{22,2} + f_2 = 0.$$

- Differentiation the first by X_1 and the second by X_2 leads to

$$\sigma_{11,11} + \sigma_{22,22} + 2\sigma_{12,12} + f_{1,1} + f_{2,2} = 0.$$

- Strain Compatibility equations in 2D reads:

$$2\mathcal{E}_{12,12} = \mathcal{E}_{11,22} + \mathcal{E}_{22,11}$$

- We, however, need compatibility in terms of stresses, not strains. Now we formalize the notion of two dimensions:

  Plane Stress $\sigma_{33} = 0$
  Plane Strain $\mathcal{E}_{33} = 0$

The "Plane Stress" Case

5. 2D Problems

- Here, we assume $\sigma_{33} = 0$ (but $\mathcal{E}_{33} \neq 0$ in general). So the stress-strain relationships are,

$$\mathcal{E}_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \quad \mathcal{E}_{22} = \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{11}$$
$$2\mathcal{E}_{12} = 2\frac{1+\nu}{E}\sigma_{12}, \quad \mathcal{E}_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

- Substituting this into the compatibility equations we get,

$$\begin{aligned} \implies \frac{2(1+\nu)}{E}\sigma_{12,12} &= \frac{1}{E}\left((\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + \sigma_{22})_{,11}\right) \\ &= \frac{1}{E}\left((\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22})\right) \end{aligned}$$

- Combining the two we get,

$$\frac{\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}}{1+\nu} + f_{1,1} + f_{2,2} = 0 => \boxed{\sigma_{ii,jj} + (1+\nu)f_{i,i} = 0}.$$

The "Plane Strain" Case

5. 2D Problems

- Here, we assume $\mathcal{E}_{33} = 0$ ($\sigma_{33} \neq 0$ in general). So the stress-strain relationships are simplified as,

$$\mathcal{E}_{33} = \frac{\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})}{E} = 0 \implies \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}),$$
$$\implies \mathcal{E}_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}) = \frac{1-\nu^2}{E}\sigma_{11} - \frac{\nu(1+\nu)}{E}\sigma_{22}$$
$$\implies \mathcal{E}_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}(\sigma_{11} + \sigma_{33}) = \frac{1-\nu^2}{E}\sigma_{22} - \frac{\nu(1+\nu)}{E}\sigma_{11}$$

- Substituting this into the compatibility equations we get,

$$\implies \frac{2(1+\nu)}{E}\sigma_{12,12} = \frac{1+\nu}{E}\Big(((1-\nu)\sigma_{11} - \nu\sigma_{22})_{,22} + (-\nu\sigma_{11} + (1-\nu)\sigma_{22})_{,11}\Big)$$
$$= \frac{1+\nu}{E}\left((1-\nu)(\sigma_{11,22} + \sigma_{22,11}) - \nu(\sigma_{11,11} + \sigma_{22,22})\right)$$

- Combining the two we get,

$$(1-\nu)(\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22}) + f_{1,1} + f_{2,2} = 0 => \boxed{\sigma_{ii,jj} + \frac{1}{1-\nu} f_{i,i} = 0}.$$

5.1. Airy's Stress Function

5. 2D Problems

- We can now combine both the **governing equations** and the **compatibility equations**, so we can write out the solution **fully in terms of stress only**.
- For the homogeneous case ($f_i = 0$), we have (for both plane stress and plane strain),

$$\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right)(\sigma_{11} + \sigma_{22}) = 0. \tag{3}$$

- We introduce the Airy's Stress function ϕ that simplifies the system of two PDE's into a scalar PDE by the substitutions:

$$\sigma_{11} := \frac{\partial^2 \phi}{\partial X_2^2}, \quad \sigma_{22} := \frac{\partial^2 \phi}{\partial X_1^2}, \quad \sigma_{12} := -\frac{\partial \phi}{\partial X_1 \partial X_2}.$$

(it is easily verified that this satisfies the governing equations $\sigma_{ij,j} = 0$ by definition)

- Substitution into eq. (3) leads to

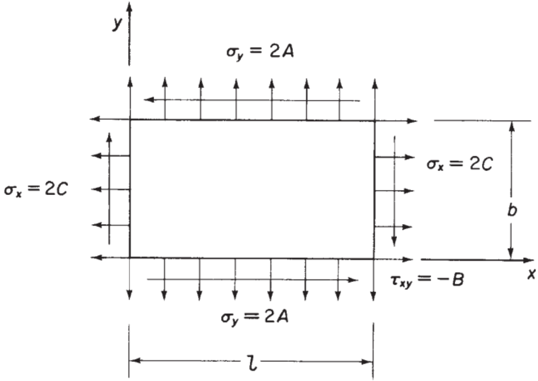
$$\phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = \left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}\right)^2 \phi = 0, \qquad \boxed{\nabla^4 \phi = 0},$$

also known as the **Biharmonic Equation**.

5.1. Airy's Stress Function: Tutorial

5. 2D Problems

- The Airy stress function can be used to solve problems with boundary loads. Consider this simple example from your textbook:

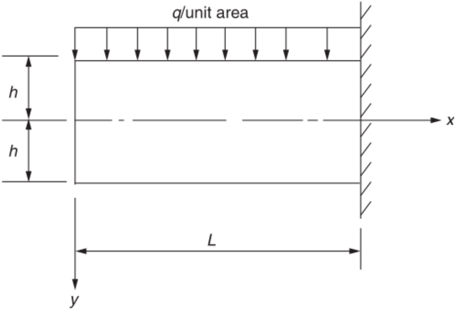


- **Airy Stress Function**: $\phi = Ax^2 + Bxy + Cy^2$

5.1. Airy's Stress Function: Tutorial

5. 2D Problems

- Consider this second example from your text book (example 2.3):



Boundary Conditions

$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad y = h$$
$$\sigma_{11} = \sigma_{12} = 0, \quad y = h$$
$$\sigma_{22} = -q, \quad y = -h$$
$$\sigma_{11} = \sigma_{22} = \sigma_{12} = 0, \quad x = 0$$
$$\int_{-h}^{h} y\sigma_{11} dy = 0, \quad x = 0$$

with a candidate Airy stress function $\phi(x, y) = Ax^2 + Bx^2y + Cy^3 + D(5x^2y^3 - y^5)$.

5.1. Airy's Stress Function

5. 2D Problems

- It may be the case that the Airy stress function doesn't meet all the boundary conditions. In this case we find a stress function that **approximately satisfies the BCs in some sense**.
- So is this completely useless? No.

St. Venant's Principle (rephrased as in Lai, Rubin, and Krempl 2010)

If ***some distribution of forces acting on a portion of the surface of a body*** *is replaced by a* ***different distribution*** *of forces acting on the* ***same portion of the body****, then the* ***effects*** *of the two different distributions on the parts of the body* ***sufficiently far removed*** *from the region of application of the forces are* ***essentially the same****, provided that the* ***two distribution of forces have the same resultant force and the same resultant couple****.*

5.1. Airy's Stress Function

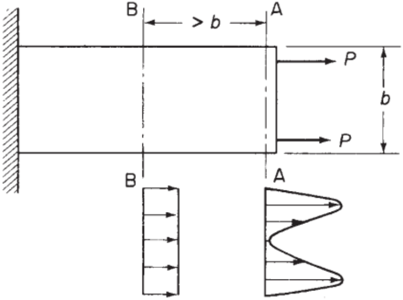
5. 2D Problems

- It may be the ca[...] the boundary conditions. In th[...] **ately satisfies the BCs in some s**[...]
- So is this comple[...]

St. Venant'[...] **Krempl 2010)**

If ***some distribution of*** [...] *is replaced by a* ***different distribution*** [...]*en the* ***effects*** *of the two different distributions* [...]*rom the region of application of the forc*[...] ***distribution of forces have the same resultan***[...]

Figure from Megson 2013



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