AS3020: Aerospace Structures

Module 8: Some Results From Buckling (V4)

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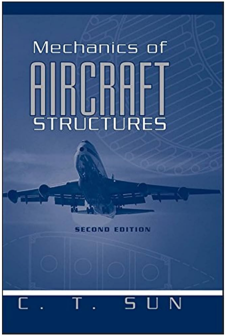
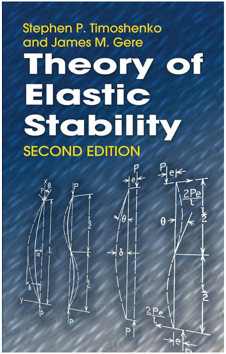


Chapter 7 in Sun (2006)



Chapter 5 in Timoshenko and Gere (2009)

1.1. Transverse Buckling I

Prismatic Structures

- The transverse buckling equation you should be exposed to thus far starts with an assumed kinematic field consisting of pure bending (assuming Kirchhoff kinematic assumptions):

$$u_1 = -(X_2 v' + X_3 w'), \quad u_2 = v, \quad u_3 = w,$$

and an assumed "dominant" compressive stress field:

$$\underset{\approx}{\sigma}_P = \begin{bmatrix} \frac{-P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with P being the applied load and A being the sectional area.

- The axial strain for the above deformation field is

$$\mathcal{E}_{11} = u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2\right),$$

where we have used the full nonlinear expression for the Lagrangian strain.

1.1. Transverse Buckling II

Prismatic Structures

- We drop the $u_{1,1}^2$ term above for the small strain case (since $u_{1,1} \gg \frac{u_{1,1}^2}{2}$) and obtain the **Von Karman Strain Expression**:

$$\mathcal{E}_{11} = u_{1,1} + \frac{u_{2,1}^2 + u_{3,1}^2}{2}.$$

- The overall stress field will be taken as the sum of the imposed field of $\sigma_{P_{11}} = -\frac{P}{A}$ and the elastic stress $\sigma_{11} = E_y \mathcal{E}_{11}$.
- The virtual work due to the stress (integrated over the section $\mathcal{S}$) is written as

$$\delta W = \int_{\mathcal{S}} \sigma_{11} \delta \mathcal{E}_{11} = \int_{\mathcal{S}} \left(-\frac{P}{A} + E_y \mathcal{E}_{11} \right) \delta \mathcal{E}_{11} = \int_{\mathcal{S}} \overbrace{-\frac{P}{A} \delta \mathcal{E}_{11}}^{-\delta \Pi} + \overbrace{E_y \mathcal{E}_{11} \delta E_{11}}^{\delta U} = \delta(U - \Pi).$$

Where we have identified $\delta \Pi$ as the load contribution and δU as the elastic contribution.

1.1. Transverse Buckling III

Prismatic Structures

- The load contribution is written as

$$\delta\Pi = \frac{P}{A}\int_{\mathcal{S}} \delta u_{1,1} + (u_{2,1}\delta u_{2,1} + u_{3,1}\delta u_{3,1}) dA$$

which, upon substitution of the above displacement field and assuming centroidal coordinate system becomes,

$$\boxed{\delta\Pi = P(v_{,1}\delta v_{,1} + w_{,1}\delta w_{,1})}.$$

- We need to integrate this along the span of the beam to obtain the overall virtual work (let us assume $P(X_1)$). This simplifies through integration by parts as:

$$\begin{aligned}\int_0^\ell \delta\Pi &= \int_0^\ell P(v_{,1}\delta v_{,1} + w_{,1}\delta w_{,1}) dX_1 \\ &= P(v_{,1}\delta v + w_{,1}\delta w)\Big|_0^\ell - \int_0^\ell \left((Pv_{,1})_{,1}\delta v + (Pw_{,1})_{,1}\delta w\right) dX_1 \\ &= -\int_0^\ell \delta \underline{V}^T \left(P\underline{V}_{,1}\right)_{,1} dX_1 + P\delta\underline{V}^T\underline{V}_{,1}\Big|_0^\ell\end{aligned}$$

1.1. Transverse Buckling IV

Prismatic Structures

- Next we consider the elastic contributions

$$\delta U = E_y \begin{bmatrix}\delta v_{,11} & \delta w_{,11}\end{bmatrix} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} v_{,11} \\ w_{,11} \end{bmatrix} = E_y \delta \underline{V}_{,11}{}^T \underline{\underline{I}}\, \underline{V}_{,11}$$

which, upon span-wise integration, becomes

$$\int_0^\ell \delta U = \delta \underline{V}_{,1}{}^T E_y \underline{\underline{I}}\, \underline{V}_{,11} \Big|_0^\ell - \delta \underline{V}^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,1} \Big|_0^\ell + \int_0^\ell \delta \underline{V}^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,11} dX_1.$$

- Putting everything together, we have

$$\int_0^\ell \delta W = \int_0^\ell \delta \underline{V}^T \left(\left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,11} + \left(P \underline{V}_{,1} \right)_{,1} \right) dX_1$$
$$- \delta \underline{V}^T \left(\left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,1} + P \underline{V}_{,1} \right) \Big|_0^\ell + \delta \underline{V}'^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right) \Big|_0^\ell.$$

By the principle of virtual work the above must be zero for equilibrium for arbitrary $\delta \underline{V}$ and $\delta \underline{V}_{,1}$.

1.1. Transverse Buckling V

Prismatic Structures

- For this equality to hold for arbitrary virtual displacements ($\delta \underline{V}$) and rotations ($\delta \underline{V}'$) each of the terms above should be equated to zero in their respective domains. So we obtain the differential equation system:

$$\left(E_y \underline{\underline{I}}\,\underline{V}_{,11}\right)_{,11} + \left(P\underline{V}_{,1}\right)_{,1} = 0, \quad X_1 \in (0,\ell)$$

$$\left(E_y \underline{\underline{I}}\,\underline{V}_{,11}\right)_{,1} + P\underline{V}_{,1} = 0, \quad \text{(OR)} \quad \underline{V} = \text{specified}, \quad X_1 \in \{0,\ell\}$$

$$E_y \underline{\underline{I}}\,\underline{V}_{,11} = 0, \quad \text{(OR)} \quad \underline{V}' = \text{specified}, \quad X_1 \in \{0,\ell\}.$$

  Note that this assumes that no other form of external load is applied.

- Considering the $\underline{e_2}$ deformation in the symmetric case ($I_{23} = 0$), the above simplifies to

$$\boxed{E_y I_{33} v'''' + P v'' = 0},$$

  which is the familiar Euler equation for buckling.

- Refer to ch. 7 in Sun (2006) for the different cases of boundary conditions herein.

1.2. Torsional Buckling I

Prismatic Structures

- For the transverse case, we started with a pure bending kinematics and derived the transverse load due to compression using the work done. **What happens when we also account for torsion?**
- Here, the kinematic deformation field is

$$u_1 = \theta_{,1}\psi, \quad u_2 = -X_3\theta, \quad u_3 = X_2\theta,$$

  where $\theta(X_1)$ is the twisting angle (we allow this to be a general function of X_1), and $\psi(X_2, X_3)$ is the St-Venant warping function (see Module 5).
- The axial strain (under Von Karman simplification, as before) is:

$$\mathcal{E}_{11} = u_{1,1} + \frac{1}{2}(u_{2,1}^2 + u_{3,1}^2) = \psi\theta_{,11} + \frac{X_2^2 + X_3^2}{2}\theta_{,1}^2.$$

- The shear strains (we only write out the linear strains) are

$$\gamma_{12} = (\psi_{,2} - X_3)\theta_{,1}, \quad \eta_{13} = (\psi_{,3} + X_2)\theta_{,1}.$$

1.2. Torsional Buckling II

Prismatic Structures

- Following the same process as above, the virtual work under an axial imposed stress field of $\sigma_{11} = -\frac{P}{A}$ is written as

$$\delta W = \int_{\mathcal{S}} (-\frac{P}{A} + E_y \mathcal{E}_{11})\delta\mathcal{E}_{11} + G\gamma_{12}\delta\gamma_{12} + G\gamma_{13}\delta\gamma_{13}$$
$$= \int_{\mathcal{S}} \overbrace{-\frac{P}{A}\delta\mathcal{E}_{11}}^{-\delta\Pi} + \overbrace{E_y\mathcal{E}_{11}\delta\mathcal{E}_{11} + G\gamma_{12}\delta\gamma_{12} + G\gamma_{13}\delta\gamma_{13}}^{\delta U} = \delta(U - \Pi).$$

- Note that unlike the bending case, we have considered shear contributions also here.
- The load contribution is written as

$$\delta\Pi = \frac{P}{A}\int_{\mathcal{S}} \psi\delta\theta_{,11} + (X_2^2 + X_3^2)\theta_{,1}\delta\theta_{,1} dA = \frac{P}{A}\delta\theta_{,11}\cancel{\int_{\mathcal{S}} \psi dA} + P\frac{I_{11}}{A}\theta_{,1}\delta\theta_{,1},$$

  where we have canceled out $\int_{\mathcal{S}} \psi dA$ because net displacement due to warping is zero. I_{11} is the polar second moment of area.
- We next consider the integral of $\delta\Pi$ and integrate it by parts to write:

$$\int_0^\ell \delta\Pi = \int_0^\ell \frac{PI_{11}}{A}\theta_{,1}\delta\theta_{,1} dX_1 = \frac{PI_{11}}{A}\theta_{,1}\delta\theta\Big|_0^\ell - \int_0^\ell \left(\frac{PI_{11}}{A}\theta_{,1}\right)_{,1} dX_1.$$

1.2. Torsional Buckling III

Prismatic Structures

- Coming to the elastic contributions, this simplifies as

$$\begin{aligned}\delta U &= \int_{\mathcal{S}} E_y \psi^2 \theta_{,11}\delta\theta_{,11} + G\left((\psi_{,2} - X_3)^2 + (\psi_{,3} + X_2)^2\right)\theta_{,1}\delta\theta_{,1} \\ &= \int_{\mathcal{S}} E_y \psi^2 dA\theta_{,11}\delta\theta_{,11} + \int_{\mathcal{S}} G(\underbrace{\psi_{,2}^2 + \psi_{,3}^2}_{-\cancel{\int_{\mathcal{S}}\psi\nabla^2\psi} + \int_{\partial\mathcal{S}}\psi\underline{\nabla}\psi\cdot\underline{n}} + X_2^2 + X_3^2 + 2X_2\psi_{,3} - 2X_3\psi_{,2})dA \\ &= \overbrace{\int_{\mathcal{S}} E_y\psi^2 dA}^{:=E_yC_w}\theta_{,11}\delta\theta_{,11} + \overbrace{G\left(I_{11} + \int_{\mathcal{S}} X_2\psi_{,3} - X_3\psi_{,2}dA\right)}^{:=GJ}\theta_{,1}\delta\theta_{,1} \\ &= E_yC_w\theta_{,11}\delta\theta_{,11} + GJ\theta_{,1}\delta\theta_{,1}.\end{aligned}$$

(Recall the governing equations and boundary conditions for ψ from Module 5)

- Here, C_w is defined as the **warping constant** of the section, and is a property of the section (like area, second moment, etc.).

1.2. Torsional Buckling IV

Prismatic Structures

- Integrating δU over the span of the beam leads to

$$\int_0^\ell \delta U = E_y C_w \theta_{,11} \delta\theta_{,1}\Big|_0^\ell - \left((E_y C_w \theta_{,11})_{,1} - GJ\theta_{,1}\right)\delta\theta\Big|_0^\ell + \int_0^\ell \left((E_y C_w \theta_{,11})_{,11} - (GJ\theta_{,1})_{,1}\right)\delta\theta dX_1$$

- Now we put everything together.

$$\int_0^\ell \delta W = \int_0^\ell \left((E_y C_w \theta_{,11})_{,11} - (GJ\theta_{,1})_{,1} + \left(\frac{PI_{11}}{A}\theta_{,1}\right)_{,1}\right)\delta\theta dX_1 + \left[E_y C_w \theta_{,11}\delta\theta_{,1} - \left((E_y C_w \theta_{,11})_{,1} - GJ\theta_{,1} + \frac{PI_{11}}{A}\theta_{,1}\right)\delta\theta\right]\Big|_0^\ell$$

1.2. Torsional Buckling V

Prismatic Structures

- The results in the following PDE governing torsional deformation under axial load:

$$\left(E_y C_w \theta_{,11}\right)_{,11} + \left(\left(\frac{P I_{11}}{A} - GJ\right)\theta_{,1}\right)_{,1} = 0, \qquad X_1 \in (0, \ell)$$

$$E_y C_w \theta_{,11} = 0, \quad \text{(OR)} \quad \theta_{,1} = \text{specified}, \quad X_1 \in \{0, \ell\}$$

$$\left(E_y C_w \theta_{,11}\right)_{,1} + \left(\frac{P I_{11}}{A} - GJ\right)\theta_{,1} = 0, \quad \text{(OR)} \quad \theta = \text{specified}, \quad X_1 \in \{0, \ell\}.$$

- For constant parameters (uniform prismatic beam) the governing equation is written as:

$$\boxed{E_y C_w \theta_{,1111} + \left(\frac{P I_{11}}{A} - GJ\right)\theta_{,11} = 0},$$

  which is a Sturm-Liouville equation of the form

$$\theta_{,1111} + p^2 \theta_{,11} = 0, \qquad p^2 = \frac{\frac{P I_{11}}{A} - GJ}{E_y C_w}$$

  which is mathematically identical to the transverse buckling equation.

- Even boundary conditions are very similar, so the solution procedure will follow exactly the same process.

1.2. Torsional Buckling: Alternative Derivation through Shear Flow Arguments I

Prismatic Structures

- It is helpful to use a more "practical" derivation to understand the $E_y C_w$ term better. We shall revert to shear flow for this - it turns out that the $E_y C_w$ term is due to the shear flow variations "induced" by the axial stress field that the warping function induces.
- From the consideration of "pure twist", the twisting moment is written as

$$M_{twist} = GJ\theta_{,1}$$

as per the notation introduced in Module 5. Recall that we had $J = I_{11} - \frac{1}{2}\int_{\partial\mathcal{S}} \frac{\partial \psi^2}{\partial n} d\ell$ for the torsion constant.
- We ignored the straight stresses σ_{11} in Module 5 since $\int_{\mathcal{S}} \sigma_{11} dA = 0$ and we argued that σ_{11} will be small. Although small, the variations are sufficient to induce additional shear flow, so let us consider it now.
- Using linear elasticity we have

$$\sigma_{11} = E_y \mathcal{E}_{11} = E_y \theta_{,11} \psi.$$

1.2. Torsional Buckling: Alternative Derivation through Shear Flow Arguments II

Prismatic Structures

- The shear flow that the section develops is written as (same procedure as we followed in Modules 4 and 5):

$$\frac{dq}{ds} + t\sigma_{11,1} = 0 \implies q_{warp}(s) - q_0 = -\int_0^s t\sigma_{11,1} ds = -E_y \theta_{,111} \int_0^s t\psi ds.$$

- Let us restrict our discussions to open sections with $q_0 = 0$ (our running integral starts from a free tip). Recall that our analysis in Module 5 led to:
  $\psi(s) = -2A_{Os}(s) = -\int_0^s p(s)ds$.
- The twisting moment that the warping shear flow $q_{warp}(s)$ leads to is written as the full integral (denoted $\int (\cdot)\, ds$)

$$M_{warp} = \int p(s) q_{warp}(s) ds = \int \left(-\frac{d\psi}{ds}\right) \left(\int_0^s -E_y \theta_{,111} t\psi(z) dz\right) ds,$$

where we have invoked $p(s) = -\frac{d\psi}{ds}$.

1.2. Torsional Buckling: Alternative Derivation through Shear Flow Arguments III

Prismatic Structures

- Applying integration by parts we have

$$M_{warp} = E_y\theta_{,11}\int \frac{d}{ds}\left(\cancel{\psi(s)\int_0^s t\psi(z)dz}\right) - t\psi^2(s)ds = -E_y\underbrace{\left(\int t\psi^2(s)ds\right)}_{C_w}\theta_{,11}$$

$$\implies \boxed{M_{warp} = -E_y C_w \theta_{,11}}.$$

  The constant C_w is called the **warping constant** and is a sectional property (much like the warping function itself, torsion constant J, area A, second moments, etc.).

- Now we have a contribution from pure twist, $GJ\theta_{,1}$, and a contribution from warping suppression, $-E_yC_w\theta_{,q11}$. Adding these both should give us the externally applied total moment:

$$\boxed{M_{tot} = GJ\theta_{,1} - E_yC_w\theta_{,111}}.$$

- For constant M_{tot}, the above can be solved to obtain $\theta(X_1)$. **This is the generalized equation governing torsional kinematics**.

1.2. Torsional Buckling: Alternative Derivation through Shear Flow Arguments IV

Prismatic Structures

- Under the presence of axial compression, $\frac{dM_{tot}}{dX_1} = m_P = -\frac{PI_{11}}{A}\theta_{,11}$. Incorporating this into the equation yields:

$$\boxed{E_y C_w \theta_{,1111} + \left(P\frac{I_{11}}{A} - GJ\right)\theta_{,11} = 0},$$

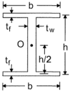
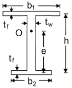
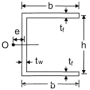
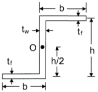
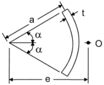
  which is identical to the equation we derived earlier.
- **This version of the derivation is only to provide you an intuitive understanding of the extra term through shear flow. This does not mean that the $E_y C_w$ term is restricted to the thin walled open section case. Since the term follows from the more formal virtual work principle, this is quite general and we can compute the $E_y C_w$ term for any arbitrary solid section/closed section too:**

$$E_y C_w = \int_{\mathcal{S}} E_y \psi^2 dA.$$

1.2. Torsional Buckling

Prismatic Structures

- The torsion constant J and warping constant C_w for common thin-walled sections are shown in the table here.

	$J = \frac{2bt_f^3 + ht_w^3}{3}$ $C_w = \frac{t_f h^2 b^3}{24}$
	$e = h\frac{b_1^3}{b_1^3 + b_2^3}$ $J = \frac{(b_1 + b_2)t_f^3 + ht_w^3}{3}$ $C_w = \frac{t_f h^2}{12}\frac{b_1^3 b_2^3}{b_1^3 + b_2^3}$
	$e = \frac{3b^2 t_f}{6bt_f + ht_w}$ $J = \frac{2bt_f^3 + ht_w^3}{3}$ $C_w = \frac{t_f b^3 h^2}{12}\frac{3bt_f + 2ht_w}{6bt_f + ht_w}$
	$J = \frac{2bt_f^3 + ht_w^3}{3}$ $C_w = \frac{b^3 h^2}{12(2b + h)^2} \times [2t_f(b^2 + bh + h^2) + 3t_w bh]$
	$e = 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$ $J = \frac{2a\alpha t^3}{3}$ $C_w = \frac{2ta^5}{3} \times \left[\alpha^3 - \frac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$

[a] O is shear center.

Table 7.2 from Sun (2006)

1.2.1. Torsional Buckling: The Simply Supported Case I

Prismatic Structures

- The simplest example is the simply supported beam under axial compression. Transverse buckling leads to:

$$\boxed{P_{cr-tr,n} = n^2 \frac{\pi^2 E_y I_p}{\ell^2}}$$

where I_p is a principal second moment (i.e., eigenvalue of the $\underline{\underline{I}}$ matrix).

- We have already seen how the torsion buckling problem is mathematically identical to the flexural buckling problem, so the critical load can directly be written as,

$$P_{cr-tw,n}\frac{I_{11}}{A} - GJ = n^2 \frac{\pi^2 E_y C_w}{\ell^2} \implies \boxed{P_{cr-tw,n} = GJ\frac{A}{I_{11}} + n^2 \frac{A}{I_{11}}\frac{\pi^2 E_y C_w}{\ell^2}}.$$

- We now have **two different critical load estimates**!
  - When P exceeds $P_{cr-tr,n}$, then the beam **buckles transversely**, or <u>bends</u>.
  - When P exceeds $P_{cr-tw,n}$, then the beam **twists**, or <u>undergoes torsion deformation</u>.
- A real designer must account for both. Let us set $n = 1$ and find when the twist-buckling will occur at a lower load than transverse buckling:

$$P_{cr-tw,1} \leq P_{cr-tr,1} \implies \boxed{GJ \leq \frac{\pi^2 E_y}{\ell^2}\left(\frac{I_{11}}{A} I_p - C_w\right)}.$$

1.2.1. Torsional Buckling: The Simply Supported Case II

- The RHS in the above represents a limiting value for the torsional rigidity of open sections. As it is, recall that $J \sim \mathcal{O}(t^3)$ for open thin walled sections, so it is not very difficult to meet the above condition in the open section case. (It is a little more difficult for closed sections)
- In practice, we find that the warping constant C_w is small/close to zero for **thin walled rectangular sections meeting at a single point** like the following.

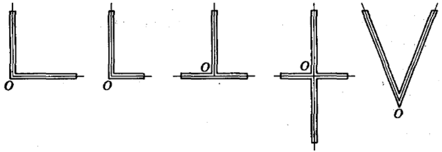


Fig. 5-5 from Timoshenko and Gere (2009)

1.2.1. Torsional Buckling: The Simply Supported Case III

Prismatic Structures

- So the condition can be simplified to

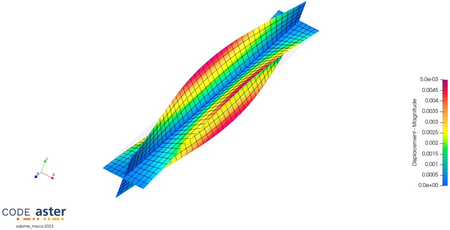
$$\boxed{GJ \leq \frac{I_{11}}{A} P_{cr-tr,1}}$$

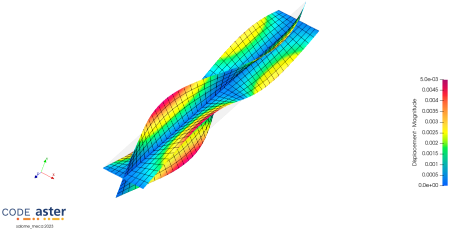
where $P_{cr-tr,1} = \frac{\pi^2 E_y I_p}{\ell^2}$ is the Euler critical load.

- This form of the equation is valid even for other boundary conditions - you just substitute the correct critical load.
- Furthermore, substituting $C_w = 0$ in the critical load expression leads to $\boxed{P_{cr-tw,n} = GJ\frac{A}{I_{11}}}$, i.e., **the (first) critical load is independent of the length of the beam!**
(Note that the $C_w = 0$ approximation no longer holds for the higher modes)

1.2.2. Torsional Buckling: Deformation Fields for a "Cross" Section

Prismatic Structures

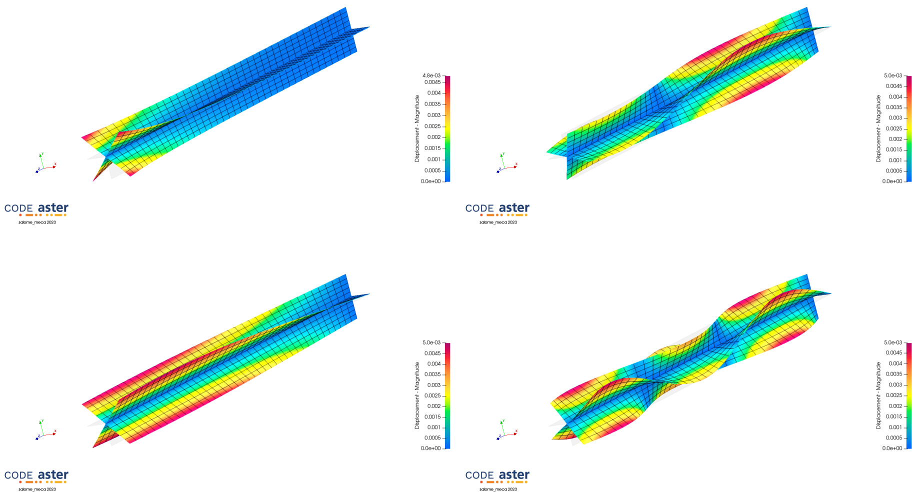




*First Two Buckling Modes in the **Simply Supported Condition***

1.2.2. Torsional Buckling: Deformation Fields for a "Cross" Section

Prismatic Structures



First Four Buckling Modes in the ***Cantilevered Condition***

2.1. Buckling of Plates I

Planar Structures

- Let us consider a plate on the $\underline{e_1} - \underline{e_2}$ plane. Invoking Kirchhoff assumptions (zero shear strain, just like we did with beams), the displacement field can be written as

$$u_1 = -X_3 w_{,1}, \quad u_2 = -X_3 w_{,2}, \quad u_3 = w.$$

- The relevant (Von Karman) strains are,

$$\begin{aligned}
\mathcal{E}_{11} &= u_{1,1} + \frac{u_{3,1}^2}{2} &&= -X_3 w_{,11} + \frac{w_{,1}^2}{2}\\
\mathcal{E}_{22} &= u_{2,2} + \frac{u_{3,2}^2}{2} &&= -X_3 w_{,22} + \frac{w_{,2}^2}{2}\\
\gamma_{12} &= u_{1,2} + u_{2,1} + u_{3,1}u_{3,2} &&= -X_3 2 w_{,12} + w_{,1}w_{,2}.
\end{aligned}$$

- The overall stress field is written as

$$\sigma_{11} = \frac{N_{11}}{t} + \frac{E_y}{1-\nu^2}(\mathcal{E}_{11} + \nu\mathcal{E}_{22}), \quad \sigma_{22} = \frac{N_{22}}{t} + \frac{E_y}{1-\nu^2}(\mathcal{E}_{22} + \nu\mathcal{E}_{11}),$$

$$\sigma_{12} = \frac{N_{12}}{t} + G\gamma_{12},$$

where N_1, N_2, N_{12} are all loads per unit length.

2.1. Buckling of Plates II

Planar Structures

- Skipping a few steps, we have the load contribution to the work coming purely from the combination of the imposed

$$\delta\Pi = N_{ij}w_{,i}\delta w_{,j}.$$

- Integrating this over the whole plate domain $\mathcal{P}$ we have,

$$\int_{\mathcal{P}} \delta\Pi = \int_{\mathcal{P}} (N_{ij}w_{,i}\delta w)_{,j} - (N_{ij}w_{,i})_{,j}\delta w dA = -\int_{\mathcal{P}} (N_{ij}w_{,i})_{,j}\delta w dA + \int_{\partial\mathcal{P}} N_{ij}w_{,i}n_j\delta w d\ell.$$

- The elastic contributions read:

$$\begin{aligned}\delta U =& \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{E_y}{1-\nu^2}\left((\mathcal{E}_{11}+\nu\mathcal{E}_{22})\delta\mathcal{E}_{11} + (\mathcal{E}_{22}+\nu\mathcal{E}_{11})\delta\mathcal{E}_{22}\right) + G\gamma_{12}\delta\gamma_{12} dX_3 \\ =& \frac{E_y}{1-\nu^2}\left(\int_{-\frac{t}{2}}^{\frac{t}{2}} X_3^2 dX_3\right)\left((w_{,11}+\nu w_{,22})\delta w_{,11} + (w_{,22}+\nu w_{,11})\delta w_{,22}\right) \\ &+ \frac{2E_y}{1+\nu}\left(\int_{-\frac{t}{2}}^{\frac{t}{2}} X_3^2 dX_3\right) w_{,12}\delta w_{,12},\end{aligned}$$

2.1. Buckling of Plates III

Planar Structures

which simplify to yield

$$\begin{aligned}\delta U &= \frac{E_y t^3}{12(1-\nu^2)}\left((w_{,11}+\nu w_{,22})\delta w_{,11} + (w_{,22}+\nu w_{,11})\delta w_{,22}\right) + 2\frac{E_y t^3}{12(1+\nu)} w_{,12}\delta w_{,12}\\ &= D\left((w_{,11}+\nu w_{,22})\delta w_{,11} + (w_{,22}+\nu w_{,11})\delta w_{,22}\right) + 2D(1-\nu) w_{,12}\delta w_{,12}.\end{aligned}$$

- We integrate this over $\mathcal{P}$ to obtain

$$\begin{aligned}\int_{\mathcal{P}} \delta U = &\int_{\mathcal{P}} \left((Dw_{,11}+\nu Dw_{,22})_{,11} + (Dw_{,22}+\nu Dw_{,11})_{,22} + 2(D(1-\nu)w_{,12})_{,12}\right)\delta w\\ &+ \int_{\partial\mathcal{P}} D(w_{,11}+\nu w_{,22})n_1\delta w_{,1} + D(w_{,22}+\nu w_{,11})n_2\delta w_{,2} + D(1-\nu)w_{,12}(n_1\delta w_{,2}+n_2\delta w_{,1})\\ &- \int_{\partial\mathcal{P}} \Big((D(w_{,11}+\nu w_{,22}))_{,1} n_1 + (D(w_{,22}+\nu w_{,11}))_{,2} n_2\\ &\qquad + (D(1-\nu)w_{,12})_{,1} n_2 + (D(1-\nu)w_{,12})_{,2} n_1 \Big)\delta w,\end{aligned}$$

where n_1, n_2 are the components of the outward pointing boundary normal vector ($\underline{n} = n_1\underline{e_1} + n_2\underline{e_2}$).

2.1. Buckling of Plates IV

Planar Structures

- For constant D and ν this simplifies to:

$$\int_{\mathcal{P}} \delta U = \int_{\mathcal{P}} D(w_{,1111} + w_{,2222} + 2w_{,1122})\delta w$$
$$- \int_{\partial\mathcal{P}} D(w_{,111}n_1 + w_{,112}n_2 + w_{,122}n_1 + w_{,222}n_2)\delta w$$
$$+ \int_{\partial\mathcal{P}} D((w_{,11} + \nu w_{,22})n_1 + (1-\nu)w_{,12}n_2)\delta w_{,1} + D((w_{,22} + \nu w_{,11})n_2 + (1-\nu)w_{,12}n_1)\delta w_{,2},$$

  where we have used symmetry to split the terms related to $w_{,12}\delta w_{,12}$.

- Using indicial notation the above can be expressed as

$$\int_{\mathcal{P}} \delta U = \int_{\mathcal{P}} Dw_{,iijj}\delta w - \int_{\partial\mathcal{P}} Dw_{,iij}n_j\delta w$$
$$+ \int_{\partial\mathcal{P}} Dw_{ij}n_j\delta w_{,i} + \nu D((w_{,22}n_1 - w_{,12}n_2)\delta w_{,1} + (w_{,11}n_2 - w_{,12}n_1)\delta w_{,2})$$

2.1. Buckling of Plates V

Planar Structures

- Now we are ready to write down the combined principle of virtual work: $\delta W = \delta U - \delta \Pi = 0$:

$$\int_{\mathcal{P}} \left(D w_{iijj} + (N_{ij} w_{,i})_{,j}\right) \delta w dA - \int_{\partial\mathcal{P}} \left(D w_{,iij} + N_{ij} w_{,i}\right) n_j \delta w$$
$$+ \int_{\partial\mathcal{P}} D((w_{,11} + \nu w_{,22}) n_1 + (1-\nu) w_{,12} n_2) \delta w_{,1}$$
$$+ \int_{\partial\mathcal{P}} D((w_{,22} + \nu w_{,11}) n_2 + (1-\nu) w_{,12} n_1) \delta w_{,2},$$

which can be interpreted in differential form (for constant N_{ij}) as:

$$
\begin{array}{rrl}
 & D\nabla^4 w + N_{11} w_{,11} + N_{22} w_{,22} + 2N_{12} w_{,12} = 0, & (X_2, X_3) \in \mathcal{P} \\
(\mathrm{I}) & (D(\nabla^2 w)_{,j} + N_{ij} w_{,i}) n_j = 0, & \\
(\mathrm{I}) & (\mathrm{OR}) \qquad w = \text{specified}, & (X_2, X_3) \in \partial\mathcal{P} \\
(\mathrm{II}) & D((w_{,11} + \nu w_{,22}) n_1 + (1-\nu) w_{,12} n_2) = 0, & \\
(\mathrm{II}) & (\mathrm{OR}) \qquad w_{,1} = \text{specified}, & (X_2, X_3) \in \partial\mathcal{P} \\
(\mathrm{III}) & D((w_{,22} + \nu w_{,11}) n_2 + (1-\nu) w_{,12} n_1) = 0, & \\
(\mathrm{III}) & (\mathrm{OR}) \qquad w_{,2} = \text{specified}, & (X_2, X_3) \in \partial\mathcal{P}.
\end{array}
$$

- Now we may consider different cases as we see fit.

2.1. Buckling of Plates I

Planar Structures

- Let us just illustrate the case of a rectangular plate simply supported on all sides.
- The domain and boundaries are written as:

$$\begin{aligned}
\mathcal{P} &= \left\{(X_1, X_2)\middle| X_1 \in (0,a)\,, \quad X_2 \in (0,b)\right\},\\
\partial\mathcal{P}_1 &= \left\{(X_1, X_2)\middle| X_1 \in (0,a)\,, X_2 = 0\right\}, \quad \underset{\sim}{n} = (0,-1),\\
\partial\mathcal{P}_2 &= \left\{(X_1, X_2)\middle| X_1 = a, \quad X_2 \in (0,b)\right\}, \quad \underset{\sim}{n} = (1,0),\\
\partial\mathcal{P}_3 &= \left\{(X_1, X_2)\middle| X_1 \in (0,a)\,, X_2 = b\right\}, \quad \underset{\sim}{n} = (0,1)\\
\partial\mathcal{P}_4 &= \left\{(X_1, X_2)\middle| X_1 = 0, \quad X_2 \in (0,b)\right\}, \quad \underset{\sim}{n} = (-1,0),\\
\partial\mathcal{P} &= \partial\mathcal{P}_1 \cup \partial\mathcal{P}_2 \cup \partial\mathcal{P}_3 \cup \partial\mathcal{P}_4.
\end{aligned}$$

- We specify $w = 0$ on all the boundaries and leave $w_{,i}$ unspecified there (so the corresponding work conjugate will be set to zero).

2.1. Buckling of Plates II

Planar Structures

- The boundary conditions can, hereby, be written as:

$$\begin{aligned}
\partial\mathcal{P}_1 &: w = 0, \quad D(1-\nu)w_{,12} &&= 0, \quad D(w_{,22} + \nu w_{,11}) &&= 0,\\
\partial\mathcal{P}_2 &: w = 0, \quad D(w_{,11} + \nu w_{,22}) &&= 0, \quad D(1-\nu)w_{,12} &&= 0,\\
\partial\mathcal{P}_3 &: w = 0, \quad D(1-\nu)w_{,12} &&= 0, \quad D(w_{,22} + \nu w_{,11}) &&= 0,\\
\partial\mathcal{P}_4 &: w = 0, \quad D(w_{,11} + \nu w_{,22}) &&= 0, \quad D(1-\nu)w_{,12} &&= 0.
\end{aligned}$$

- For the case with $N_{22} = N_{12} = 0$, $N_{11} \neq 0$, the governing equations are

$$D\nabla^4 w + N_{11} w_{,11} = 0.$$

Check Megson 2013 for the solutions of this.

References I

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