AS3020: Aerospace Structures

Module 8: Some Results From Buckling (V2)

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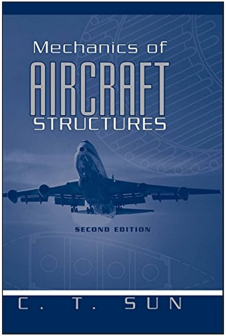
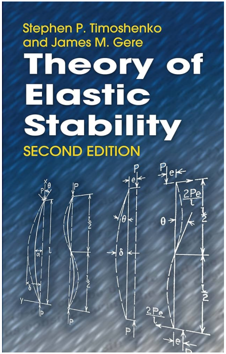
Table of Contents



Chapter 7 in Sun (2006)



Chapter 5 in Timoshenko and Gere (2009)

1.1. Transverse Buckling I

Prismatic Structures

- The transverse buckling equation you should be exposed to thus far starts with an assumed kinematic field consisting of pure bending (assuming Kirchhoff kinematic assumptions):

$$u_1 = -(X_2 v' + X_3 w'), \quad u_2 = v, \quad u_3 = w,$$

  and an assumed "dominant" compressive stress field:

$$\underset{\approx}{\sigma}_P = \begin{bmatrix} \frac{-P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

  with P being the applied load and A being the sectional area.
- The axial strain for the above deformation field is

$$\mathcal{E}_{11} = u_{1,1} + \frac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2\right),$$

  where we have used the full nonlinear expression for the Lagrangian strain.

1.1. Transverse Buckling II

Prismatic Structures

- We drop the $u_{1,1}^2$ term above for the small strain case (since $u_{1,1} \gg \frac{u_{1,1}^2}{2}$) and obtain the **Von Karman Strain Expression**:

$$\mathcal{E}_{11} = u_{1,1} + \frac{u_{2,1}^2 + u_{3,1}^2}{2}.$$

- The overall stress field will be taken as the sum of the imposed field of $\sigma_{P_{11}} = -\frac{P}{A}$ and the elastic stress $\sigma_{11} = E_y \mathcal{E}_{11}$.
- The virtual work due to the stress is written as

$$\delta W = \sigma_{11} \delta\mathcal{E}_{11} = \left(-\frac{P}{A} + E_y \mathcal{E}_{11}\right) \delta\mathcal{E}_{11} = \overbrace{-\frac{P}{A}\delta\mathcal{E}_{11}}^{-\delta\Pi} + \overbrace{E_y \mathcal{E}_{11} \delta E_{11}}^{\delta U} = \delta(U - \Pi).$$

Where we have identified $\delta\Pi$ as the load contribution and δU as the elastic contribution.

1.1. Transverse Buckling III

Prismatic Structures

- The load contribution is written as

$$\delta\Pi = \frac{P}{A}\int_{\mathcal{S}} \delta u_{1,1} + (u_{2,1}\delta u_{2,1} + u_{3,1}\delta u_{3,1}) dA$$

which, upon substitution of the above displacement field and assuming centroidal coordinate system becomes,

$$\boxed{\delta\Pi = P(v_{,1}\delta v_{,1} + w_{,1}\delta w_{,1})}.$$

- We need to integrate this along the span of the beam to obtain the overall virtual work (let us assume $P(X_1)$). This simplifies through integration by parts as:

$$\begin{aligned}
\int_0^\ell \delta\Pi &= \int_0^\ell P(v_{,1}\delta v_{,1} + w_{,1}\delta w_{,1}) dX_1 \\
&= P(v_{,1}\delta v + w_{,1}\delta w)\Big|_0^\ell - \int_0^\ell \left((Pv_{,1})_{,1}\delta v + (Pw_{,1})_{,1}\delta w\right) dX_1 \\
&= -\int_0^\ell \delta\underline{V}^T \left(P\underline{V}_{,1}\right)_{,1} dX_1 + P\delta\underline{V}^T\underline{V}_{,1}\Big|_0^\ell
\end{aligned}$$

1.1. Transverse Buckling IV

Prismatic Structures

- Next we consider the elastic contributions

$$\delta U = E_y \begin{bmatrix} \delta v_{,11} & \delta w_{,11} \end{bmatrix} \begin{bmatrix} I_{33} & I_{23} \\ I_{23} & I_{22} \end{bmatrix} \begin{bmatrix} v_{,11} \\ w_{,11} \end{bmatrix} = E_y \delta \underline{V}_{,11}{}^T \underline{\underline{I}}\, \underline{V}_{,11}$$

which, upon span-wise integration, becomes

$$\int_0^\ell \delta U = \delta \underline{V}_{,1}{}^T E_y \underline{\underline{I}}\, \underline{V}_{,11} \Big|_0^\ell - \delta \underline{V}^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,1} \Big|_0^\ell + \int_0^\ell \delta \underline{V}^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,11} dX_1.$$

- Putting everything together, we have

$$\begin{aligned} \int_0^\ell \delta W = \int_0^\ell \delta \underline{V}^T \left(\left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,11} + \left(P \underline{V}_{,1} \right)_{,1} \right) dX_1 \\ - \delta \underline{V}^T \left(\left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right)_{,1} + P \underline{V}_{,1} \right) \Big|_0^\ell + \delta \underline{V}'^T \left(E_y \underline{\underline{I}}\, \underline{V}_{,11} \right) \Big|_0^\ell . \end{aligned}$$

By the principle of virtual work the above must be zero for equilibrium for arbitrary $\delta \underline{V}$ and $\delta \underline{V}_{,1}$.

1.1. Transverse Buckling V

Prismatic Structures

- For this equality to hold for arbitrary virtual displacements ($\delta \underline{V}$) and rotations ($\delta \underline{V}'$) each of the terms above should be equated to zero in their respective domains. So we obtain the differential equation system:

$$\left(E_y \underline{\underline{I}}\, \underline{V}_{,11}\right)_{,11} + \left(P \underline{V}_{,1}\right)_{,1} = 0, \quad X_1 \in (0, \ell)$$

$$\left(E_y \underline{\underline{I}}\, \underline{V}_{,11}\right)_{,1} + P \underline{V}_{,1} = 0, \quad \text{(OR)} \quad \underline{V} = \text{specified}, \quad X_1 \in \{0, \ell\}$$

$$E_y \underline{\underline{I}}\, \underline{V}_{,11} = 0, \quad \text{(OR)} \quad \underline{V}' = \text{specified}, \quad X_1 \in \{0, \ell\}.$$

  Note that this assumes that no other form of external load is applied.

- Considering the $\underline{e_2}$ deformation in the symmetric case ($I_{23} = 0$), the above simplifies to

$$\boxed{E_y I_{33} v'''' + P v'' = 0},$$

  which is the familiar Euler equation for buckling.

- Refer to ch. 7 in Sun (2006) for the different cases of boundary conditions herein.

1.2. Torsional Buckling

Prismatic Structures

- For the transverse case, we started with a pure bending kinematics and derived the transverse load due to compression using the work done. **What happens when we also account for torsion?**
- Here, the kinematic deformation field is

$$u_1 = \theta_{,1}\psi, \quad u_2 = -X_3\theta, \quad u_3 = X_2\theta,$$

where $\theta(X_1)$ is the twisting angle (we allow this to be a general function of X_1), and $\psi(X_2, X_3)$ is the St-Venant warping function (see Module 5).
- The axial strain (under Von Karman simplification, as before) is:

$$\mathcal{E}_{11} = u_{1,1} + \frac{1}{2}(u_{2,1}^2 + u_{3,1}^2) = \psi\theta_{,11} + \frac{X_2^2 + X_3^2}{2}\theta_{,1}^2.$$

- Under an axial stress field of $\sigma_{11} = \frac{-P}{A}$, the work done by the load (per unit length) simplifies as

$$\Pi = -\sigma_{11}\mathcal{E}_{11} = \frac{P}{A}\int_{\mathcal{S}} \psi\theta_{,11} + \frac{X_2^2 + X_3^2}{2}\theta_{,1}^2 dA = \frac{P}{A}\theta_{,11}\cancel{\int_{\mathcal{S}} \psi dA} + P\frac{I_{11}}{2A}\theta_{,1}^2,$$

where we have canceled out $\int_{\mathcal{S}} \psi dA$ because net displacement due to warping is zero. I_{11} is the polar second moment of area.
- We stationarize the work potential and obtain the twist contribution from the load:

$$\delta\Pi = P\frac{I_{11}}{2A}\delta(\theta_{,1}^2) = -P\frac{I_{11}}{A}\theta_{,11}\delta\theta \implies m_p = \frac{\partial(\delta\Pi)}{\partial(\delta\theta)} = -P\frac{I_{11}}{A}\theta_{,11}.$$

1.2. Torsional Buckling I

Prismatic Structures

- From the consideration of "pure twist", the twisting moment is written as

$$M_{twist} = GJ\theta_{,1}$$

as per the notation introduced in Module 5. Recall that we had $J = I_{11} - \frac{1}{2}\int_{\partial\mathcal{S}} \frac{\partial\psi^2}{\partial n} d\ell$ for the torsion constant.

- We ignored the straight stresses σ_{11} in Module 5 since $\int_{\mathcal{S}} \sigma_{11} dA = 0$ and we argued that σ_{11} will be small. Although small, the variations are sufficient to induce additional shear flow, so let us consider it now.
- Using linear elasticity we have

$$\sigma_{11} = E_y\mathcal{E}_{11} = E_y\theta_{,11}\psi.$$

- The shear flow that the section develops is written as (same procedure as we followed in Modules 4 and 5):

$$\frac{dq}{ds} + t\sigma_{11,1} = 0 \implies q_{warp}(s) - q_0 = -\int_0^s t\sigma_{11,1} ds = -E_y\theta_{,111}\int_0^s t\psi ds.$$

1.2. Torsional Buckling II

Prismatic Structures

- Let us restrict our discussions to open sections with $q_0 = 0$ (our running integral starts from a free tip). Recall that our analysis in Module 5 led to:
  $\psi(s) = -2A_{Os}(s) = -\int_0^s p(s)ds$.
- The twisting moment that the warping shear flow $q_{warp}(s)$ leads to is written as the full integral (denoted $\int (\cdot)\, ds$)

$$M_{warp} = \int p(s) q_{warp}(s) ds = \int \left(-\frac{d\psi}{ds}\right) \left(\int_0^s -E_y \theta_{,111} t\psi(z) dz\right) ds,$$

  where we have invoked $p(s) = -\frac{d\psi}{ds}$.
- Applying integration by parts we have

$$M_{warp} = E_y \theta_{,11} \int \frac{d}{ds} \cancel{\left(\psi(s) \int_0^s t\psi(z) dz\right)} - t\psi^2(s) ds = -E_y \underbrace{\left(\int t\psi^2(s) ds\right)}_{C_w} \theta_{,11}$$

$$\implies \boxed{M_{warp} = -E_y C_w \theta_{,11}}.$$

  The constant C_w is called the **warping constant** and is a sectional property (much like the warping function itself, torsion constant J, area A, second moments, etc.).

1.2. Torsional Buckling III

Prismatic Structures

- Now we have a contribution from pure twist, $GJ\theta_{,1}$, and a contribution from warping suppression, $-E_yC_w\theta_{,q11}$. Adding these both should give us the externally applied total moment:

$$\boxed{M_{tot} = GJ\theta_{,1} - E_yC_w\theta_{,111}}.$$

- For constant M_{tot}, the above can be solved to obtain $\theta(X_1)$. **This is the generalized equation governing torsional kinematics**.
- Under the presence of axial compression, $\frac{dM_{tot}}{dX_1} = m_P = -\frac{PI_{11}}{A}\theta_{,11}$. Incorporating this into the equation yields:

$$\boxed{E_yC_w\theta_{,1111} + \left(P\frac{I_{11}}{A} - GJ\right)\theta_{,11} = 0},$$

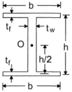
which is Sturm-Liouville problem of the form:

$$\theta_{,1111} + p^2\theta_{,11} = 0, \quad p^2 = \frac{P\frac{I_{11}}{A} - GJ}{E_yC_w}.$$

1.2. Torsional Buckling

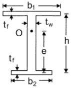
Prismatic Structures

- The torsion constant J and warping constant C_w can be derived for any given thin walled section. The expressions for common sections are shown in the table here.



$$J = \frac{2bt_f^3 + ht_w^3}{3}$$

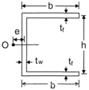
$$C_w = \frac{t_f h^2 b^3}{24}$$



$$e = h\frac{b_1^3}{b_1^3 + b_2^3}$$

$$J = \frac{(b_1 + b_2)t_f^3 + ht_w^3}{3}$$

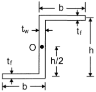
$$C_w = \frac{t_f h^2}{12}\frac{b_1^3 b_2^3}{b_1^3 + b_2^3}$$



$$e = \frac{3b^2 t_f}{6bt_f + ht_w}$$

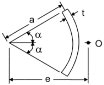
$$J = \frac{2bt_f^3 + ht_w^3}{3}$$

$$C_w = \frac{t_f b^3 h^2}{12}\frac{3bt_f + 2ht_w}{6bt_f + ht_w}$$



$$J = \frac{2bt_f^3 + ht_w^3}{3}$$

$$C_w = \frac{b^3 h^2}{12(2b+h)^2} \times [2t_f(b^2 + bh + h^2) + 3t_w bh]$$



$$e = 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$$

$$J = \frac{2a\alpha t^3}{3}$$

$$C_w = \frac{2ta^5}{3} \times \left[\alpha^3 - \frac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$$

[a] O is shear center.

Table 7.2 from Sun (2006)

1.2.1. Torsional Buckling: The Simply Supported Case I

Prismatic Structures

- The simplest example is the simply supported beam under axial compression. Transverse buckling leads to:

$$\boxed{P_{cr-tr,n} = n^2 \frac{\pi^2 E_y I_p}{\ell^2}}$$

where I_p is a principal second moment (i.e., eigenvalue of the $\underline{\underline{I}}$ matrix).

- Under pure torsion, we obviously set $\theta = 0$ (zero twist) at the ends. But, since it is a free end, σ_{11} has to be zero everywhere. That is, $\sigma_{11} = E_y \theta_{,11} \psi(X_2, X_3) = 0$. Since ψ is non-trivial, $\theta_{,11} = 0$ represents the boundary condition.
- This is mathematically very similar to the moment-free boundary condition which is $E_y I w'' = 0$. **This is just a coincidence**, but this renders the two problems mathematically identical, yielding,

$$P_{cr-tw,n} \frac{I_{11}}{A} - GJ = n^2 \frac{\pi^2 E_y C_w}{\ell^2} \implies \boxed{P_{cr-tw,n} = GA \frac{J}{I_{11}} + n^2 \frac{A}{I_{11}} \frac{\pi^2 E_y C_w}{\ell^2}}.$$

- We now have **two different critical load estimates**!
  - When P exceeds $P_{cr-tr,n}$, then the beam **buckles transversely**, or <u>bends</u>.
  - When P exceeds $P_{cr-tw,n}$, then the beam **twists**, or <u>undergoes torsion deformation</u>.

1.2.1. Torsional Buckling: The Simply Supported Case II

Prismatic Structures

- A real designer must account for both. Let us set $n = 1$ and find when the twist-buckling will occur at a lower load than transverse buckling:

$$P_{cr-tw,1} \leq P_{cr-tr,1} \implies \boxed{GJ \leq \frac{\pi^2 E_y}{\ell^2} \left(\frac{I_{11}}{A} I_p - C_w \right)}.$$

- The RHS in the above represents a limiting value for the torsional rigidity of open sections. As it is, recall that $J \sim \mathcal{O}(t^3)$ for thin walled sections, so it is not very difficult to meet the above condition.
- In practice, we find that the warping constant C_w is small/close to zero for thin walled rectangular sections meeting at a point like the following.

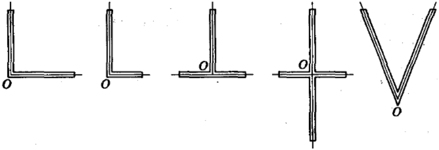


Fig. 5-5 from Timoshenko and Gere (2009)

1.2.1. Torsional Buckling: The Simply Supported Case III

Prismatic Structures

- So the condition can be simplified to

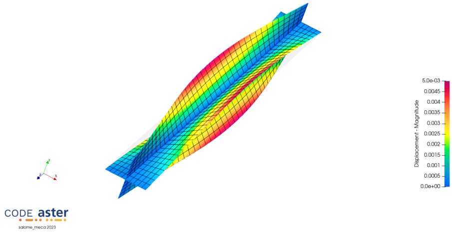
$$\boxed{GJ \leq \frac{I_{11}}{A} P_{cr-tr,1}}$$

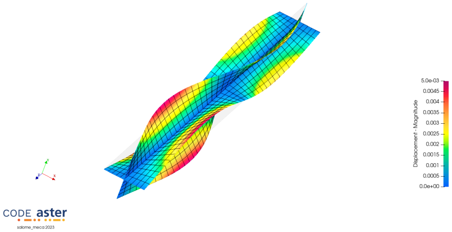
where $P_{cr-tr,1} = \frac{\pi^2 E_y I_p}{\ell^2}$ is the Euler critical load.

- This form of the equation is valid even for other boundary conditions - you just substitute the correct critical load.
- Furthermore, substituting $C_w = 0$ in the critical load expression leads to $\boxed{P_{cr-tw,n} = GA\frac{J}{I_{11}}}$, i.e., **the (first) critical load is independent of the length of the beam!**
  (Note that the $C_w = 0$ approximation no longer holds for the higher modes)

1.2.2. Torsional Buckling: Deformation Fields for a "Cross" Section

Prismatic Structures

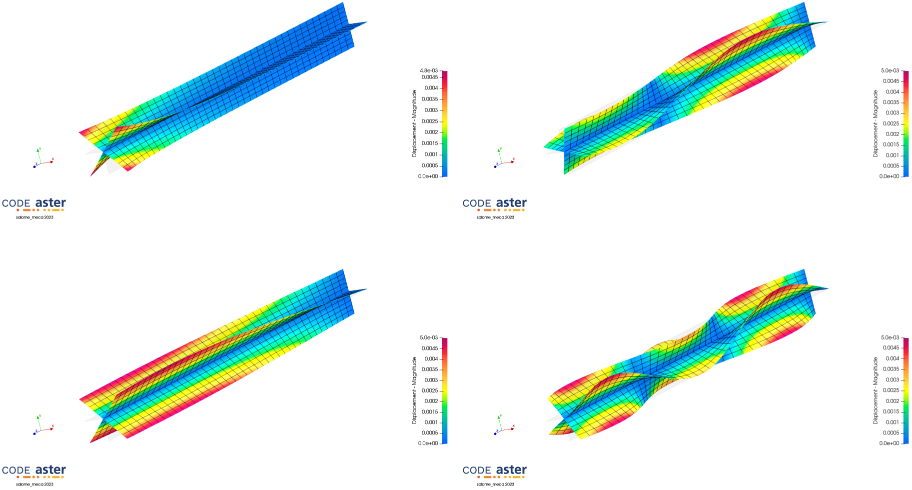




First Two Buckling Modes in the ***Simply Supported Condition***

1.2.2. Torsional Buckling: Deformation Fields for a "Cross" Section

Prismatic Structures



First Four Buckling Modes in the ***Cantilevered Condition***

2.1. Buckling of Plates

Planar Structures

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[2] Stephen P. Timoshenko and James M. Gere. Theory of Elastic Stability, Courier Corporation, June 2009. ISBN: 978-0-486-47207-2 (cit. on pp. 2, 14).