#18781: Response Synthesis of Frictional Systems under Narrow-Band Random Excitation: A Nonlinear Mode Approach Based on Time-Scale Decomposition

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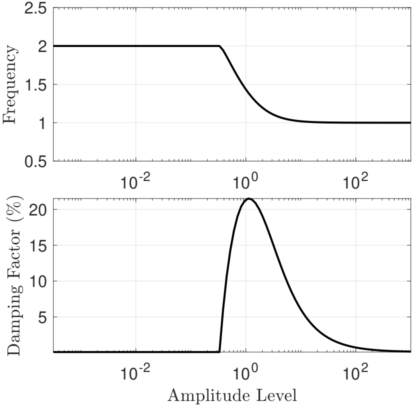
1. Introduction

- The literature on nonlinear modal analysis is reasonably mature.
- Techniques have been developed to use nonlinear modal backbones to synthesize responses.



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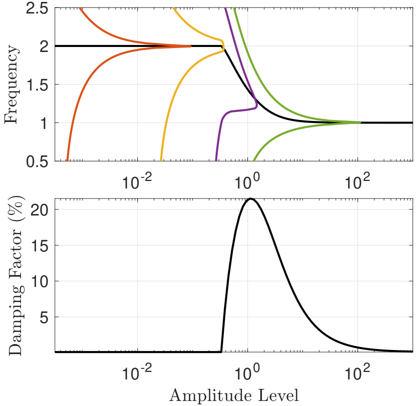

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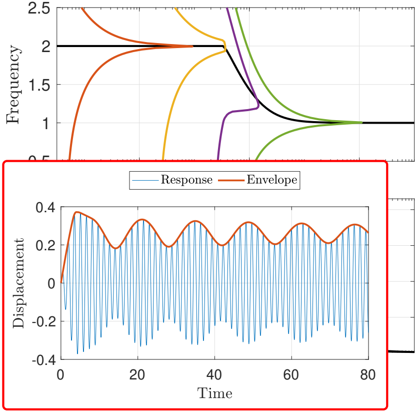

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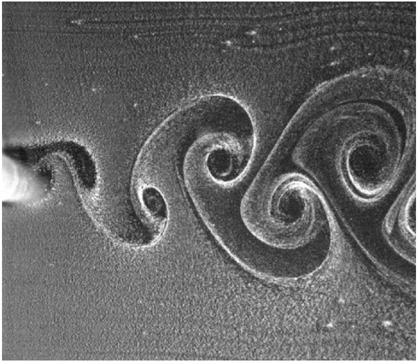
- Non-periodic excitation very common in practice.
  - Impulsive excitations
  - Broad-band random excitation
  - Narrow-band excitation[a]



Vortex-Induced Vibrations (VIV)

(Figure from Huang et al. 2019)

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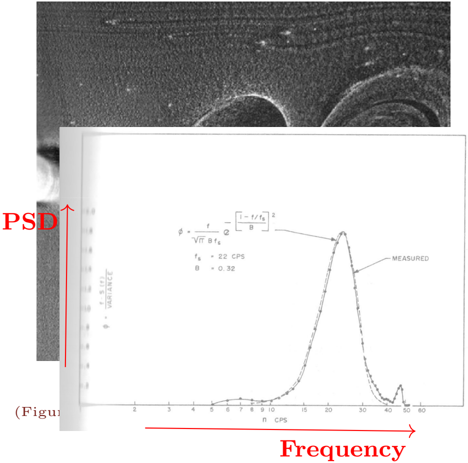

(Figu

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2. Methodology

Goals

- Given nonlinear modal characteristics, we want to synthesize **responses to near-resonant random excitation**
- Develop numerical experience and apply it to an experimental system

2.1. Nonlinear Modal Analysis and Synthesis

Methodology

Problem Setting: Numerical Model

$$\underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\underline{C}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} + \underline{f_{nl}}(\underline{u}) = \underline{f_{ex}}(t)$$

- Replacing the excitation with a velocity-proportional self-excitation term, we characterize the **effective undamped nonlinear modes**[1]:

$$\begin{aligned} \underline{\underline{M}}\,\ddot{\underline{u}} + \underline{\underline{C}}\,\dot{\underline{u}} + \underline{\underline{K}}\,\underline{u} + \underline{f_{nl}}(\underline{u}) &= 2\zeta\omega\underline{\underline{M}}\,\dot{\underline{u}} \\ \frac{\omega}{\pi}\int_0^{2\pi/\omega} \underline{u}^T\underline{\underline{M}}\,\underline{u}dt &= q^2 \end{aligned} \implies \begin{aligned} &\omega(q), \zeta(q), \underline{\psi}(q) \\ &\underline{u} \approx \tfrac{q}{2}\underline{\psi}(q)e^{\imath\omega t} + \text{c.c.}. \end{aligned}$$

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Modal Synthesis using Complexification Averaging

- Under excitation of the form $\underline{f_{ex}}(t) = \frac{\underline{f_e}}{2}e^{\imath\sigma(t)} + \text{c.c.}$, the response is written as,

$$\underline{u}(t) = \frac{\boxed{q(t)}}{2}\underline{\psi}(q)\exp\left(\imath(\sigma + \boxed{\beta(t)})\right) + \text{c.c.}$$

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"slow" amplitude

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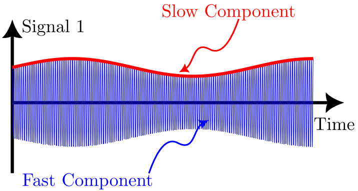
2.1. Nonlinear Modal Analysis and Synthesis

Methodology

Problem Setting: Numerical Model

Amplitude-Phase Dynamics using Complexification Averaging (CXA)



$$\dot{q} = -\zeta(q)\omega(q)q + \frac{1}{2\Omega}\Im\{\underline{\psi}(q)^H \boxed{\underline{f_e}}\, e^{-\imath\beta}\}$$

$$\dot{\beta} = -\frac{\Omega^2 - \omega(q)^2}{2\Omega} - \frac{1}{2\Omega q}\Re\{\underline{\psi}(q)^H \boxed{\underline{f_e}}\, e^{-\imath\beta}\} - \left(\boxed{\dot{\sigma}} - \Omega\right)$$

- Replacin ... on term, we characte ...

$\underline{\underline{M}}$ ... $), \underline{\psi}(q)$

... ωt + c.c..

- Under ex ... ritten as,

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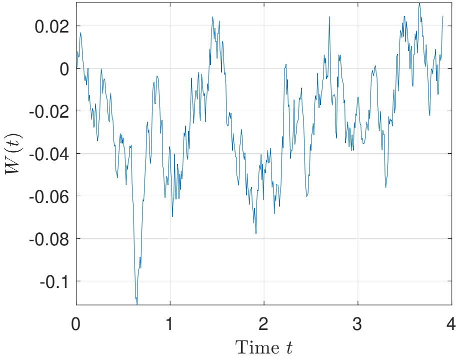
2.2. Stochastic Processes and Narrow-Band Excitation

Methodology

- A **Stochastic Process** is the response of a dynamical system under random excitation.
- The simplest setting is the **Wiener Process** $W(t)$:

$$dW = W_{t+1} - W_t = \eta, \quad \eta \sim \mathcal{N}(0,1).$$



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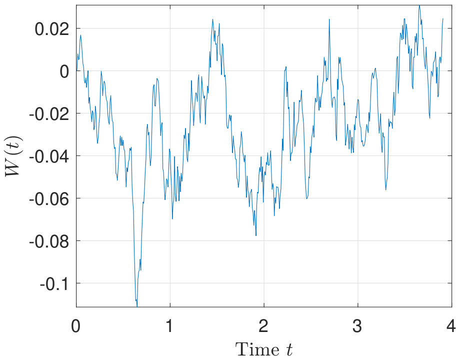

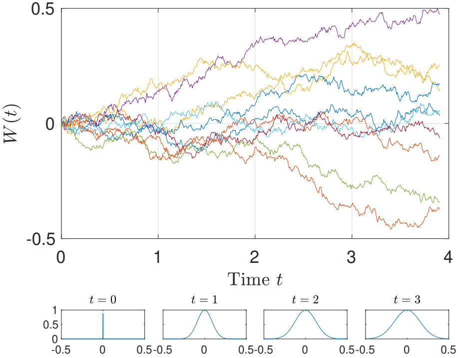

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- It has been shown[2] that a narrow-band random excitation can be expressed as

$$f(t) = F\cos(\sigma(t)), \quad d\sigma = \Omega dt + \nu dW$$

by modeling the excitation phase σ as a stochastic process.

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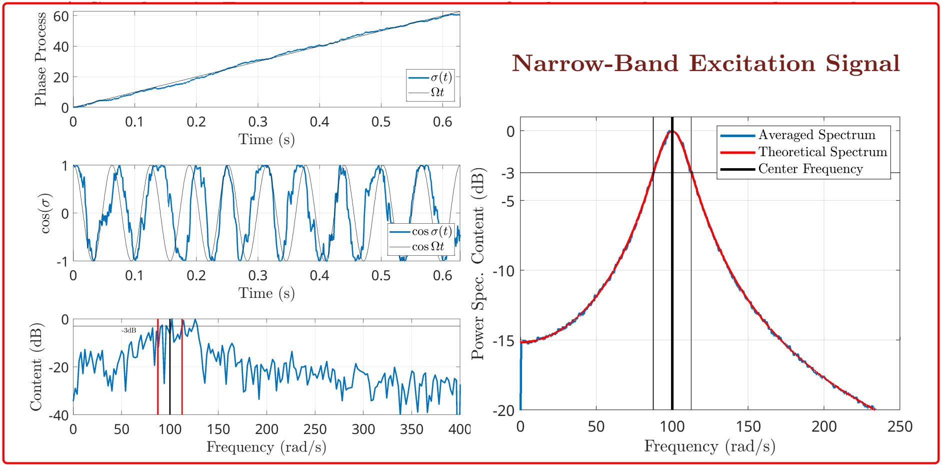

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Introducing this into the amplitude-phase dynamics leads to the SDE:

$$d\begin{bmatrix} q \\ \beta \end{bmatrix} = \begin{bmatrix} -\zeta\omega q - \frac{F}{2\Omega}\sin\beta \\ -\frac{\Omega^2-\omega^2}{2\Omega} - \frac{F}{2\Omega q}\cos\beta \end{bmatrix} dt + \begin{bmatrix} 0 \\ -\nu \end{bmatrix} dW$$

$$\boxed{d\underline{y} = \underline{f_1}(\underline{y})dt + \underline{f_2}dW}$$

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2.3. The Fokker-Planck Equation

Methodology

- Given an SDE of the form

$$\boxed{d\underline{y} = \underline{f_1}(\underline{y})dt + \underline{f_2}dW}$$

Fokker-Planck Equations provide a Partial Differential Equation governing the diffusion of the joint probability density $p(\underline{y})$:

$$\frac{\partial p}{\partial t} = \underline{\nabla} \cdot \left(\underline{f_1} p + \frac{1}{2} \underline{\nabla} \cdot \left(\underline{f_2} \underline{f_2}^T p \right) \right), \quad \text{i.e.,} \quad \boxed{\dot{p} = \underline{\nabla} \cdot \underline{J}}$$

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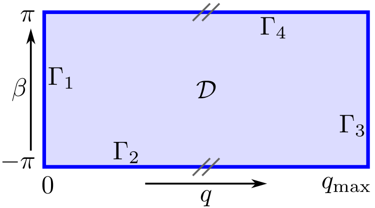
Finite-Element Implementation



Boundary Conditions

$$J_i\big|_{\Gamma_1} = 0, \forall\, i = 1, 2$$

$$p\big|_{\Gamma_2} = p\big|_{\Gamma_4};\ p\big|_{\Gamma_3} = 0$$

- Using linear finite elements, this results in

$$\boxed{\underline{\underline{M}}\,\underline{\dot{p}} = \underline{\underline{K}}\,\underline{p}}.$$

- **Stationary Solutions:** zero-eigenpair of

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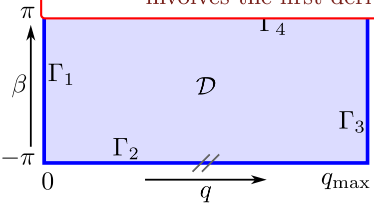
Special Note: The advective term $\underline{\nabla} \cdot (\underline{f_1}p)$ involves the first derivatives of $\omega(q), \zeta(q), \underline{\psi}(q)$



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3.1. Numerical Examples

Applications

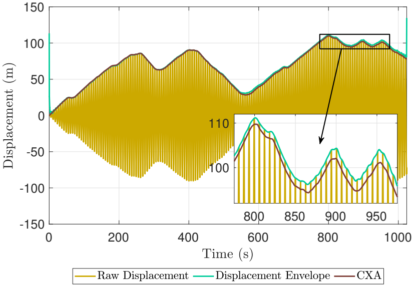
SDoF Linear Oscillator

Center frequency

Band-width

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = F\cos\sigma,$$

$$\omega_0 = 1\,\mathrm{rad/s},\quad \zeta = 0.1\,\%,\quad F = 1\,\mathrm{N},\quad \Omega = \omega_0,\quad b_w = 0.03\,\mathrm{rad/s}$$



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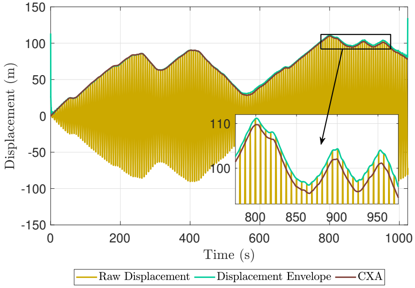

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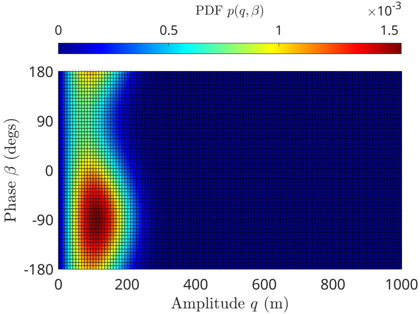

(b) PDF from Fokker Planck Equations

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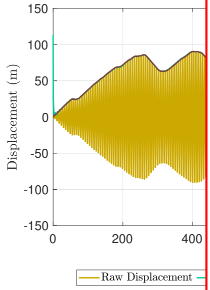

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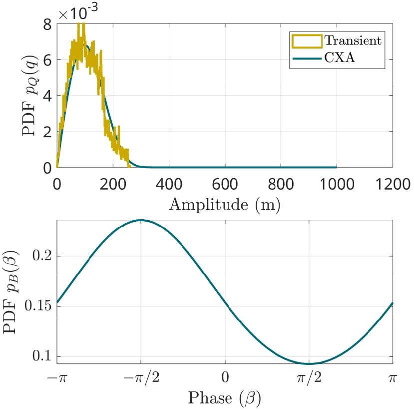

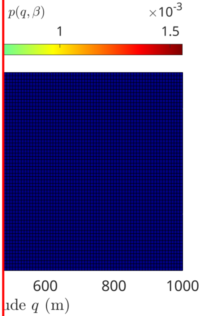

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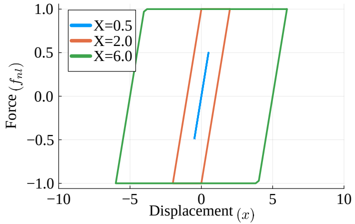

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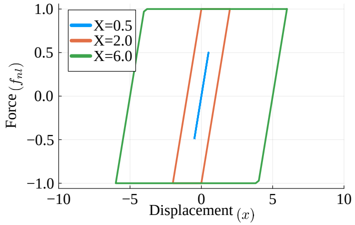
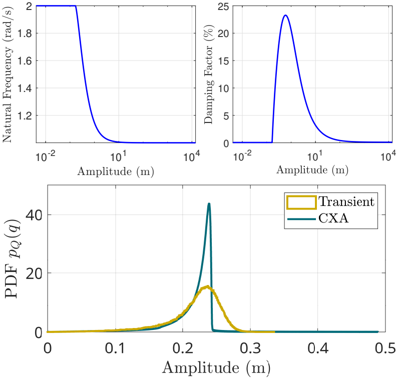


Results



Parameter Set 1

$\Omega = 1.8\,\text{rad/s},\ (k_t, f_s) = (3\,\text{N/m}, 0.5\,\text{N})$

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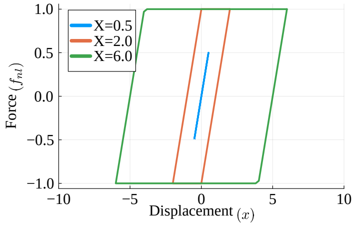

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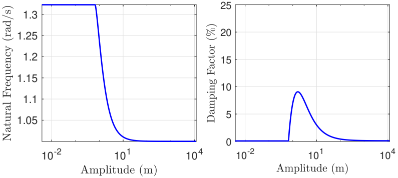

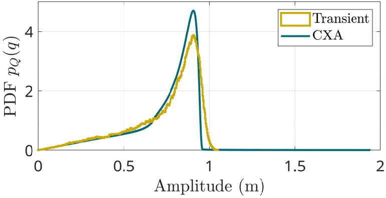

Parameter Set 2 (reduced peak damping)
$\Omega = 1.6\,\mathrm{rad/s},\ (k_t, f_s) = (2\,\mathrm{N/m}, 0.5\,\mathrm{N})$

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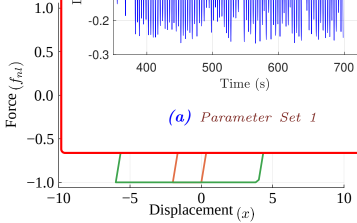
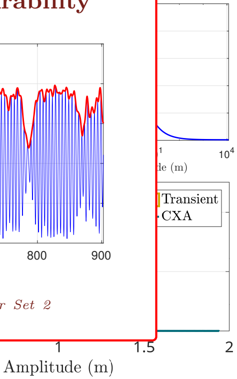
Applications

Results

SD

$\ddot{x} + 2\zeta_0$





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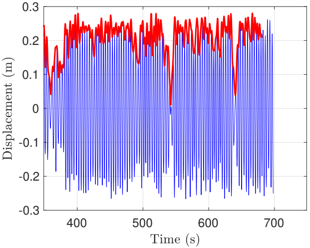
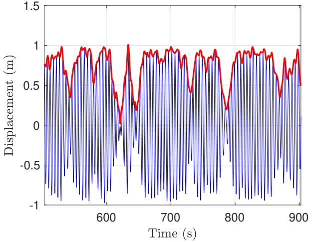
Reasoning: Validity of Time-Scale Separability



(a) *Parameter Set 1*

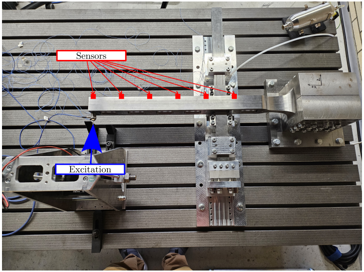


(b) *Parameter Set 2*

3.2. Experimental Test Case

Applications

The Rubbing Beam Resonator (RubBeR)
(Scheel, Weigele, and Krack 2020)



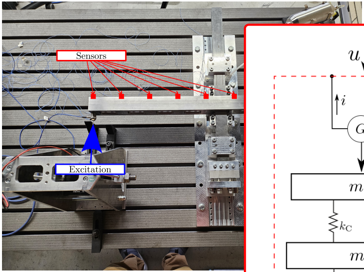
Experimental Setup

3.2. Experimental Test Case

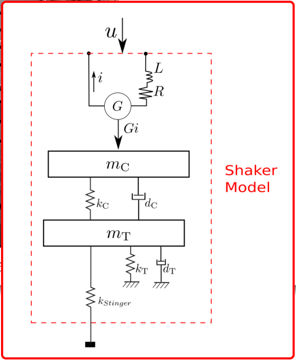
Applications

The Rubbing Beam Resonator (RubBeR)

(Scheel, Weigele, and Krack 2020)



Experimental Se

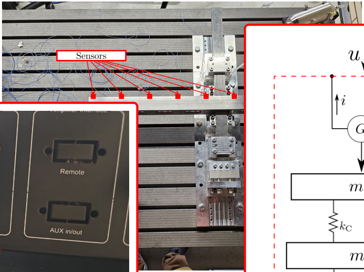


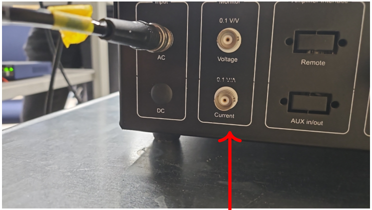
3.2. Experimental Test Case

Applications

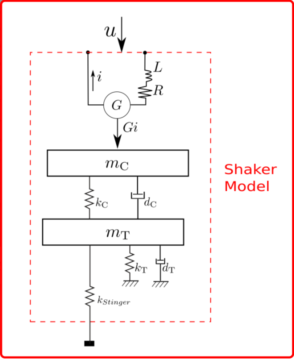
The Rubbing Beam Resonator (RubBeR)

(Scheel, Weigele, and Krack 2020)





rimental Se



Input→ Voltage $u = A\cos\sigma$,

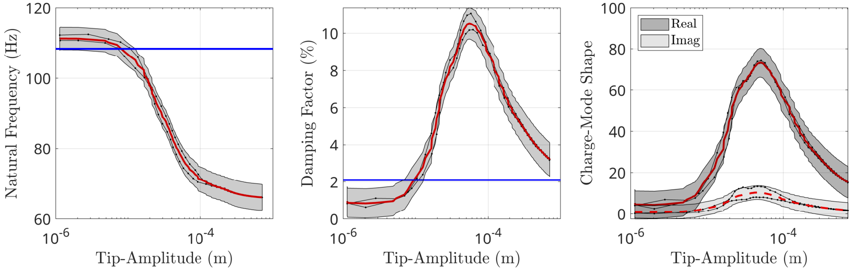
Dynamical Quantity→ Charge, Current

3.2. Experimental Test Case

Applications

The Rubbing Beam Resonator (RubBeR)

(Scheel, Weigele, and Krack 2020)

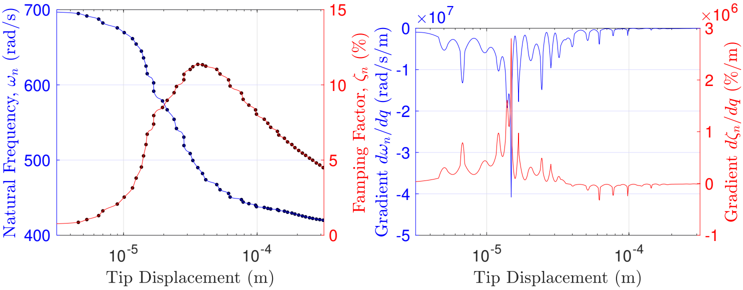


Experimentally Extracted Backbones with PLL

3.2. Experimental Test Case

Applications

Raw PCHIP



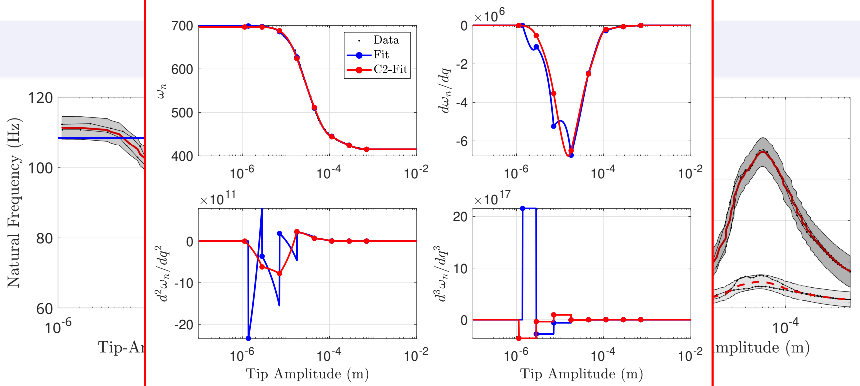
Tip-Amplitude (m) Tip-Amplitude (m) Tip-Amplitude (m)

Experimentally Extracted Backbones with PLL

3.2. Experimental Test Case

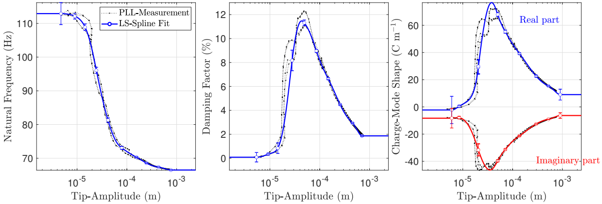
Applications

C2-Constrained Least-Square Splines



3.2. Experimental Test Case: Results

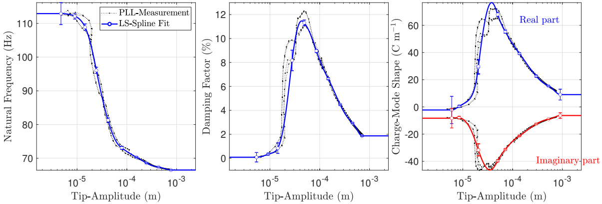
Applications



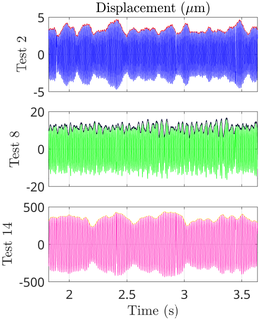
SNos.	Center Freq. (Hz)	B-Width (Hz)	Voltage (V)
1-6	110	0,0.5,1 2,4,8	0.1
7-12	90	0,0.5,1 2,4,8	1
13-18	70	0,0.5,1 2,4,8	5
19	90	0	0.5
20	90	0.5	0.5
21	90	0	2
22	90	0.5	2
23	110	30	0.1
24	110	30	1
25	70	30	5
26	70	30	1

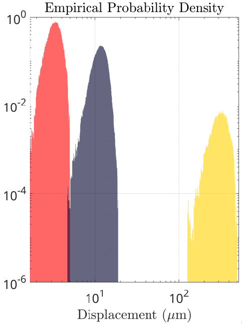
3.2. Experimental Test Case: Results

Applications



SNos.	Center Freq. (Hz)	B-Width (Hz)	Voltage (V)
1-6	110	0,0.5,1 2,4,8	0.1
7-12	90	0,0.5,1 2,4,8	1
13-18	70	0,0.5,1 2,4,8	5
19	90	0	0.5
20	90	0.5	0.5
21	90	0	2
22	90	0.5	2
23	110	30	0.1
24	110	30	1
25	70	30	5
26	70	30	1



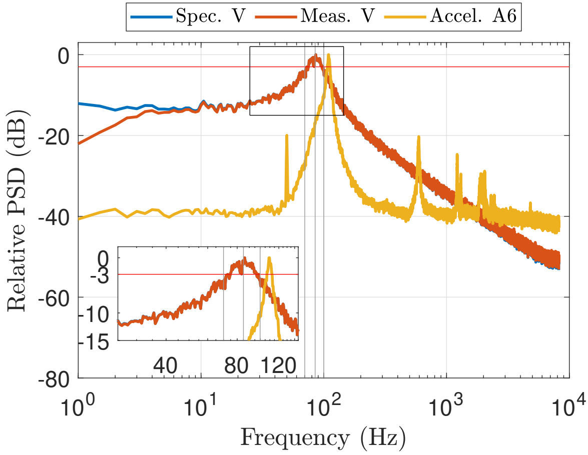


3.2. Experimental Test Case: Results

Applications

SNos.
1-6
7-12
13-18
19 20 21 22
23 24
25 26

Response Spectra

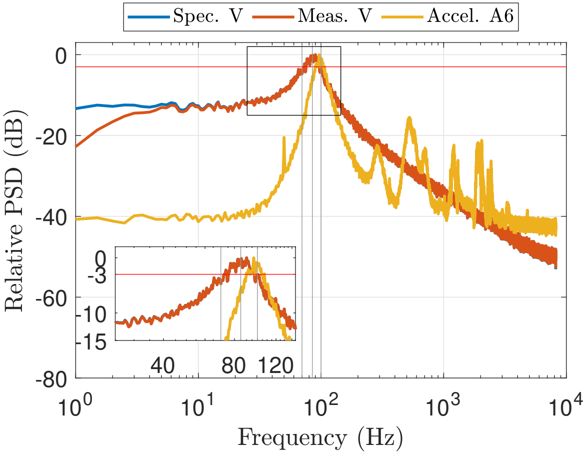


Excitation Level: 0.2 V

3.2. Experimental Test Case: Results

Applications

Response Spectra



Excitation Level: 1 V

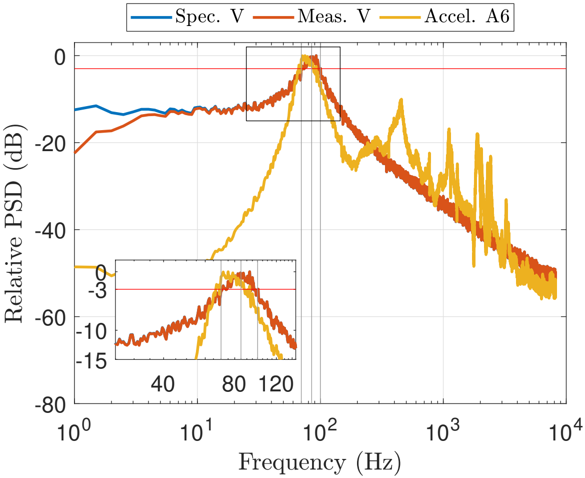
SNos.
1-6
7-12
13-18
19 20 21 22
23 24
25 26

3.2. Experimental Test Case: Results

Applica

SNos.
1-6
7-12
13-18
19
20
21
22
23
24
25
26

Response Spectra



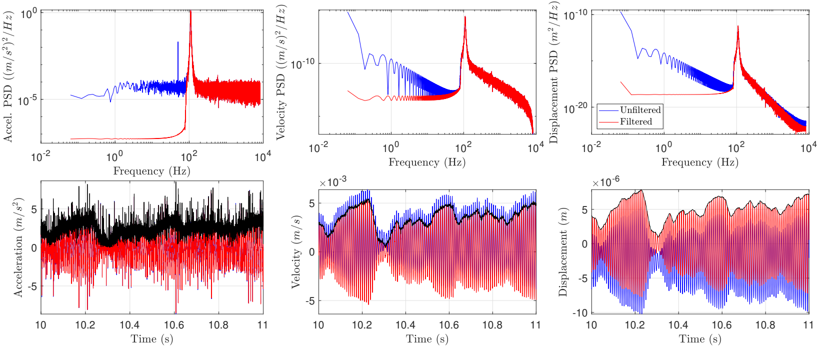
Excitation Level: 5 V

3.2. Experimental Test Case: Nonlinear Modal Synthesis

Applications

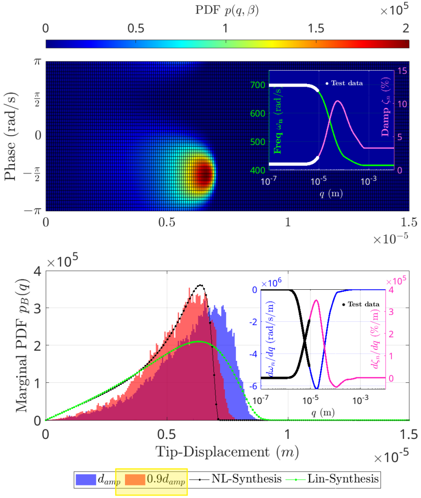
Synthesis is displacement-based:

The acceleration data needs to be (filtered and) integrated



3.2. Experimental Test Case: Nonlinear Modal Synthesis

Applications

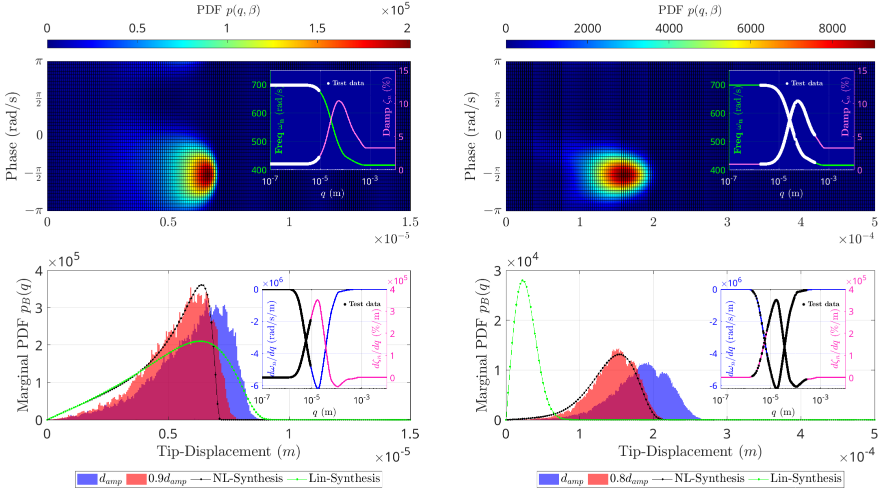


(a)

$\Omega = 110\,\mathrm{Hz}(691\,\mathrm{rad/s}),\ B_w = 5\,\mathrm{Hz}(31\,\mathrm{rad/s}),\ V = 0.1\,\mathrm{V}$

3.2. Experimental Test Case: Nonlinear Modal Synthesis

Applications

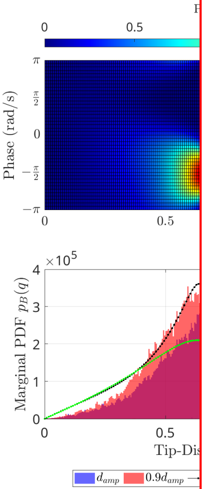


(a)
$\Omega = 110\,\text{Hz}(691\,\text{rad/s}),\ B_w = 5\,\text{Hz}(31\,\text{rad/s}),\ V = 0.1\,\text{V}$

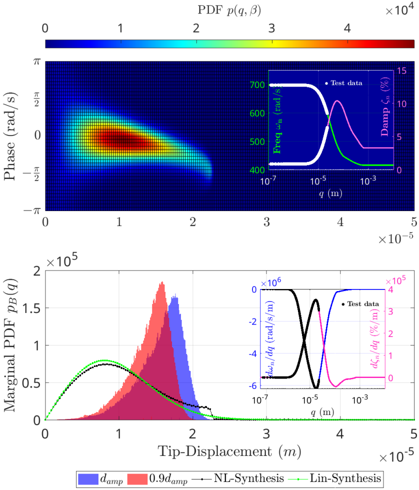
(b)
$\Omega = 70\,\text{Hz}(440\,\text{rad/s}),\ B_w = 5\,\text{Hz}(31\,\text{rad/s}),\ V = 5.0\,\text{V}$

3.2. Experimental Test Case: Nonlinear Modal Synthesis

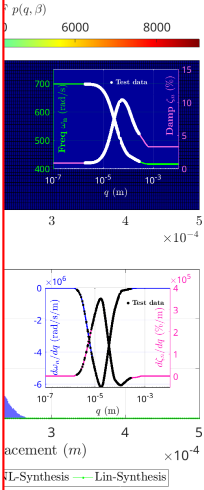
Applications



(a)
$\Omega = 110\,\text{Hz}(691\,\text{rad/s}),\ B_w$



(c) $\Omega = 90\,\text{Hz}(565\,\text{rad/s}),\ B_w = 5\,\text{Hz}(31\,\text{rad/s}),\ V = 1.0\,\text{V}$



$5\,\text{Hz}(31\,\text{rad/s}),\ V = 5.0\,\text{V}$

4. Conclusions

- It is possible to express a nonlinear system undergoing **near-resonant narrow-band excitation** as a Stochastic Differential Equation in the amplitude-phase coordinates
- Numerical results show promise for the application of the Fokker-Planck Equations for predicting the stationary solution directly: **This presents a novel use-case of Nonlinear Modal Analysis**
- The methodology seems to struggle with the presence of large damping:
  - **In the micro-slip regime, the resonance properties vary strongly**

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- The methodology seems to struggle with the presence of large damping:
  - **In the micro-slip regime, the resonance properties vary strongly**

Future Work

- The influence of nonlinear modal backbone uncertainties on the synthesized probability density (Uncertainty Propagation) may improve the applicability of the approach
- Considerations for multi-modal response regimes

References I

[1] B. J. Vickery and A. W. Clark. “Lift or Across-Wind Response of Tapered Stacks”. Journal of the Structural Division, **98**,1 (Jan. 1972), pp. 1–20. DOI: 10.1061/JSDEAG.0003103. (Visited on 07/26/2022) (cit. on pp. 2–6).

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