AS5900

Lecture 13

__Problem:__ What conditions should a given $e_{ij}(\underset{\sim}{x})$ satisfy so that $\varepsilon_{ij} = \dfrac{u_{i,j} + u_{j,i}}{2}$ for some $u_i(\underset{\sim}{x})$.

__Answer:__ $e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$

system of 3^4 equations.

$i=1, j=1, k=1, l=1$:

$$\cancel{\frac{\partial^2 e_{xx}}{\partial x^2}} + \cancel{\frac{\partial^2 e_{xx}}{\partial x^2}} - \cancel{\frac{\partial^2 e_{xx}}{\partial x^2}} - \cancel{\frac{\partial^2 e_{xx}}{\partial x^2}} = 0$$

$\Rightarrow \quad 0 = 0 \ldots$ True, but useless.

$i=1, j=1, k=1, l=2$

$$\cancel{\frac{\partial^2 e_{xx}}{\partial x \partial y}} + \cancel{\frac{\partial^2 e_{xy}}{\partial x^2}} - \cancel{\frac{\partial^2 e_{xx}}{\partial x \partial y}} - \cancel{\frac{\partial^2 e_{xy}}{\partial x^2}} = 0$$

... Trivially true.

$i = j = 1; \quad k = l = 2$

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - \frac{\partial^2 e_{xy}}{\partial x \partial y} - \frac{\partial^2 e_{xy}}{\partial x \partial y} = 0$$

$$\Rightarrow \quad \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \partial y} \quad (1)$$

Another non-trivial eqn:

$i = j = 2; \; k = l = 3$

$$\frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = 2 \frac{\partial^2 e_{yz}}{\partial y \partial z} \quad (2)$$

Yet another n-t eqn got by cyclic permutation:

$i = j = 3; \; k = l = 1$

$$\frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} = 2 \frac{\partial^2 e_{zx}}{\partial z \partial x} \quad (3)$$

Other 3 non-trivial eqns are:

$$\frac{\partial^2 e_{xx}}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right) \quad (4)$$

Obtain eqns (5) & (6) by cyclic permutation.

6 non-trivial equations only out of the 81 total compatibility conditions. Others are trivially satisfied.

How to get Eq (4)?

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$

$i = 2,\ j = 3;\ k = l = 1.$

$$e_{23,11} + e_{11,23} - e_{21,31} - e_{31,21} = 0.$$

(5): $i=3;\ j=1;\ k=l=2$ || (6): $i=1,\ j=2;\ k=l=3.$

The 6 non-trivial equations can be expressed even more succinctly:

$$\eta_{ij} \equiv \varepsilon_{ikl}\, \varepsilon_{jmp}\, e_{lp,km} = 0$$

"Incompatibility tensor".

$\eta_{ij} = \eta_{ji}$ — Symmetry. — 6 indep. η_{ij}

6 non-trivial compatibility eqns are the same as demanding the 6 conditions

$$\eta_{ij} = \varepsilon_{ikl}\, \varepsilon_{jmp}\, e_{lp,km} = 0.$$

$$\left.\begin{array}{l}\text{Try } i=1, j=2 \\ \text{H/W: You try } i=j=1.\end{array}\right\} \text{Rest follow by cyclic permutation.}$$

$i=1,\ j=2$

$$\eta_{12} = \varepsilon_{1kl}\,\varepsilon_{2mp}\,e_{lp,km} \stackrel{?}{=} 0.$$

$$\begin{matrix} 2\ 3 & 1\ 3 \\ 3\ 2 & 3\ 1 \end{matrix}$$

$$\overset{1}{\cancel{\varepsilon_{123}}}\left\{ \overset{-1}{\cancel{\varepsilon_{213}}}\, e_{33,21} + \overset{1}{\cancel{\varepsilon_{231}}}\, e_{31,23} \right\}$$

$$+$$

$$\overset{-1}{\cancel{\varepsilon_{132}}}\left\{ \overset{-1}{\cancel{\varepsilon_{213}}}\, e_{23,31} + \overset{1}{\cancel{\varepsilon_{231}}}\, e_{21,33} \right\} = 0.$$

(2)

$$\Rightarrow -e_{33,21} + e_{31,23} + e_{23,31} - e_{21,33} = 0. \quad (6)$$

Compare w/

$$\frac{\partial^2 e_{zz}}{\partial x\,\partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} \right)$$

$\eta_{12} = 0 \iff$ Eq. (6).

ε_{ij} — 6 nos

$u_i \longrightarrow$ 3 nos

$$A\,x = b$$

$$\begin{bmatrix} \boxed{A_1} \\ \boxed{\alpha A_1} \end{bmatrix}_{6\times 3} \left\{ x \right\}_{3\times 1} = \begin{bmatrix} \boxed{b_1} \\ \boxed{\alpha b_1} \end{bmatrix}_{6\times 1}.$$

I need only 3 additional condns. to be satisfied so that the 6 eqns. should have a solution — "Compatibility Condns".

This suggests that our compatibility eqns 1 – 6 are also not really independent — only 3 of them are truly independent

Job for tomorrow is to exclude 3 of these dependent eqns.