AS5900

Lecture 14

Q: Given $e_{ij}(\underset{\sim}{x})$ over a body what conditions must it satisfy so there is a u_i such that

$$e_{ij}(\underset{\sim}{x}) = \frac{u_{i,j} + u_{j,i}}{2}.$$

A: Define

$$\eta_{ij}(\underset{\sim}{x}) = \varepsilon_{ikl}\, \varepsilon_{jmp}\, e_{lp,km}.$$

If $\eta_{ij} \equiv 0$ everywhere in the body, then there is a $u_i(\underset{\sim}{x})$ corresponding to the given $e_{ij}(\underset{\sim}{x})$

A2: We $\eta_{ij} = \eta_{ji}$

So there are 6 $\eta_{ij} = 0$ eqns — "Compatibility condns."

Q3: 6 — $\varepsilon_{ij}(\underset{\sim}{x})$ — prescribed
3 — $u_i(\underset{\sim}{x})$ — to be found.

$$[A]_{6\times 3}\{x\}_{3\times 1} = \{b\}_{6\times 1}.$$

Additional condn for solution:
Eqs 4, 5, 6 should be a linear recombination of Eqs 1, 2, 3

→ 3 such condns.

How come we have 6 compat. condns, $\eta_{ij} = 0$?
We should only have 3!

A3: The η_{ij}'s are themselves not independent.
They satisfy the relation

$$\eta_{ij,j} = 0 \quad \ldots \quad 3 \text{ such eqns.}$$

Proof that $\eta_{ij,j} = 0$.

$$\eta_{ij} = \varepsilon_{ikl}\, \varepsilon_{jmp}\, e_{lp,km}$$

$$\eta_{ij,j} = \cancelto{0}{\varepsilon_{ikl,j}}\, \varepsilon_{jmp}\, e_{lp,km} + \varepsilon_{ikl}\, \cancelto{0}{\varepsilon_{jmp,j}}\, e_{lp,km} + \varepsilon_{ikl}\, \varepsilon_{jmp}\, e_{lp,kmj}$$

$$\eta_{ij,j} = \varepsilon_{ikl} \underbrace{\left\{ \varepsilon_{jmp}\, e_{lp,jmk} \right\}}_{\equiv 0}$$

$$\Rightarrow \boxed{\eta_{ij,j} = 0.}$$

3 such conditions

2.17 Say
$$e_{xx} = Ay^3$$
$$e_{yy} = Ax^3$$
$$e_{xy} = Bxy(x+y)$$
$$e_{zz} = e_{zx} = e_{zy} = 0.$$

For what A, B will we be able to find u_i?

Ans: A, B should be determined from the compat. Eqns.

$$\boxed{① \quad \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}}$$

$$6Ay + 6Ax = 2B(2x+2y)$$

has no z in it.

$$② \quad \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} = 2\frac{\partial e_{yz}}{\partial y \partial z}$$

0   0   0

Compatibility

Check that Eqs ② – ⑥ are trivially satisfied.

$6A = 4B$

$\Rightarrow 3A = 2B$

$\Rightarrow A = 2B/3.$

• finding u_i itself can be more complicated.

$$u_i^P = u_i^0 + \omega_{ij}^P x_j^P + \int_C U_{ik} dx_k$$

$$e_{ik} - x_j \left(e_{ik,j} - e_{jk,i} \right)$$

§1.9 — Curvilinear orthogonal coordinate systems.

§2.7 — Strain tensor, in polar, spherical etc.

In Cartesian,

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

$$\varepsilon_{12} = \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \frac{1}{2}$$

In other coordinates (polar, spherical)

Convert this into an index-free form.

$$\underset{\sim}{\varepsilon} = \frac{(\nabla \underset{\sim}{u}) + (\nabla \underset{\sim}{u})^T}{2}$$

Then derive expressions for $\nabla \underset{\sim}{u}$ etc in that coordinate system.

Indicial notation does not work in non-cartesian coord. systems.

§1.9.

Cartesian

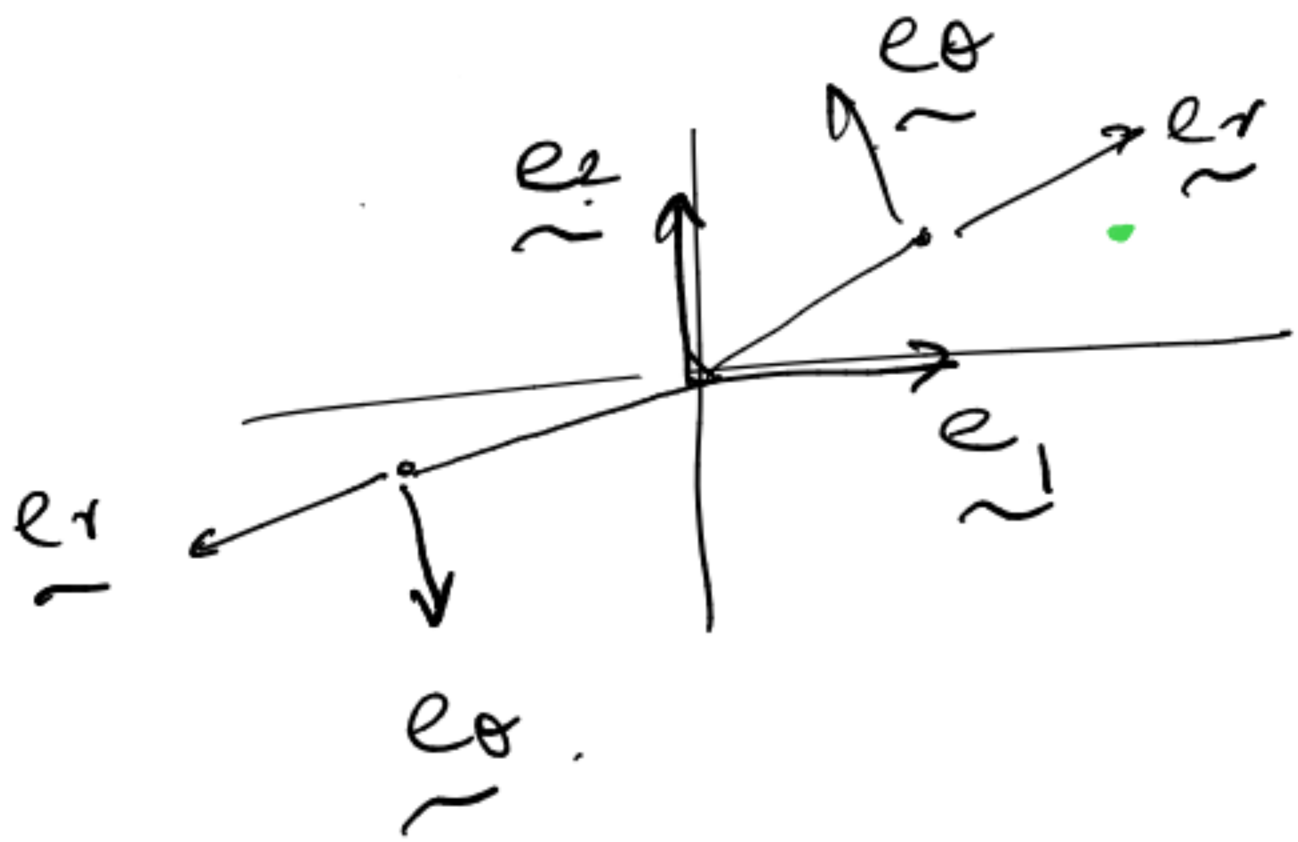
(x_1, x_2, x_3)

$(\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3})$

Another non-Cartesian

(ξ_1, ξ_2, ξ_3) (r, θ, z) in polar

$(\underset{\sim}{\hat{e}_1}, \underset{\sim}{\hat{e}_2}, \underset{\sim}{\hat{e}_3})$ $(\underset{\sim}{e_r}, e_\theta, e_z)$



Find relations between (x_i) & (ξ_i)

polar:

$x_1 = r\cos\theta$

$x_2 = r\sin\theta$

$x_3 = z$

$\xi_1 = r$

$\xi_2 = \theta$

$\xi_3 = z$.

$r = \sqrt{x_1^2 + x_2^2}$

$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$; $z = x_3$.

We next have to find $d\xi_1, d\xi_2, d\xi_3$ in terms of dx_1, dx_2, dx_3.

$$\overrightarrow{ds}$$

$$(ds)^2 = (dx_1^2) + (dx_2)^2 + (dx_3)^2$$

$$= (h_1 d\xi_1)^2 + (h_2 d\xi_2)^2 + (h_3 d\xi_3)^2$$

h_1, h_2, h_3 are called scale factors along ξ_1, ξ_2, ξ_3 directions, respectively.

Unit vectors in $\xi_1 \xi_2 \xi_3$ coord system:

unit vector along ξ_1

$$\underset{\sim}{\hat{e}_1} = \frac{1}{h_1} \frac{\partial x_k}{\partial \xi_1} \underset{\sim}{e_k}.$$

$$\underset{\sim}{\hat{e}_2} = \frac{1}{h_2} \frac{\partial x_k}{\partial \xi_2} \underset{\sim}{e_k}$$

$$\underset{\sim}{\hat{e}_3} = \frac{1}{h_3} \frac{\partial x_k}{\partial \xi_3} \underset{\sim}{e_k}.$$

To find h_1, h_2, h_3 note

$$\hat{\underset{\sim}{e}}_i \cdot \hat{\underset{\sim}{e}}_j = \delta_{ij}$$

$$\hat{\underset{\sim}{e}}_1 \cdot \hat{e}_1 = 1$$

$$\left(\frac{1}{h_1} \frac{\partial x_k}{\partial \xi_1} \underset{\sim}{e}_k\right) \cdot \left(\frac{1}{h_1} \frac{\partial x_l}{\partial \xi_1} \underset{\sim}{e}_l\right) = 1.$$

$$\Rightarrow \frac{1}{h_1^2} \frac{\partial x_k}{\partial \xi_1} \frac{\partial x_l}{\partial \xi_1} \underbrace{\cancel{\underset{\sim}{e}_k \cdot \underset{\sim}{e}_l}}_{\delta_{kl}} = 1.$$

$$\Rightarrow h_{\underset{3}{\overset{2}{1}}} = \left\{ \frac{\partial x_k}{\partial \xi_{\underset{3}{\overset{2}{1}}}} \frac{\partial x_k}{\partial \xi_{\underset{3}{\overset{2}{1}}}} \right\}^{1/2}$$