

A55900

Lecture 15

Orthogonal Cartesian (x_1, x_2, x_3)	curvilinear coordinates. <u>Orthogonal Curvilinear</u> (ξ_1, ξ_2, ξ_3)
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$$\xi_i = \xi_i(x_1, x_2, x_3)$$

e.g. in cylindrical coordinates:

$$r = \xi_1 = \sqrt{x_1^2 + x_2^2}$$

$$\theta = \xi_2 = \tan^{-1} \frac{x_2}{x_1}$$

$$z = \xi_3 = x_3$$

$$x_i = x_i(\xi_1, \xi_2, \xi_3)$$

e.g. in cyl. coordinates

$$x_1 = r \cos \theta \quad x_2 = r \sin \theta \quad x_3 = z$$

Unit vectors of the curvilinear coordinate system:

$$\hat{e}_1 = \frac{1}{h_1} \frac{\partial x_k}{\partial \xi_1} \hat{i}_k$$

Cartesian \hat{i}_k

Curvilinear unit vector

along ξ_1

$$\hat{e}_2 = \frac{1}{h_2} \frac{\partial x_k}{\partial \xi_2} \hat{i}_k$$

Cartesian \hat{j}_k

$$\hat{e}_3 = \frac{1}{h_3} \frac{\partial x_k}{\partial \xi_3} \hat{i}_k$$

h_1, h_2, h_3 — scale factors
 — scalar functions.

found by requiring $\hat{e}_1 \cdot \hat{e}_1 = 1$
 $\hat{e}_2 \cdot \hat{e}_2 = 1$
 $\hat{e}_3 \cdot \hat{e}_3 = 1$

Eg: find $\hat{e}_1, \hat{e}_2, \hat{e}_3, h_1, h_2, h_3$
for cylindrical coordinates.

$$\hat{e}_1 = \frac{1}{h_1} \frac{\partial x_k}{\partial x_1} \hat{e}_k$$

$$= \frac{1}{h_1} \left[\frac{\partial x_1}{\partial r} \hat{e}_1 + \frac{\partial x_2}{\partial r} \hat{e}_2 + \frac{\partial x_3}{\partial r} \hat{e}_3 \right]$$

$$= \frac{1}{h_1} \left[\cos \theta \hat{e}_1 + \sin \theta \hat{e}_2 \right]$$

find h_1 by requiring

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$

$$\frac{1}{h_1^2} \left[\cos^2 \theta + \sin^2 \theta \right] = 1 \Rightarrow h_1 = 1,$$

h_i always +ve.

$$\hat{e}_r = \cos \theta \hat{e}_1 + \sin \theta \hat{e}_2$$

$\hat{e}_1 = \hat{r}$
 $\hat{e}_2 = \hat{\theta}$

$$\hat{e}_2 = \frac{1}{h_2} \left[\frac{\partial x_k}{\partial \theta} \hat{e}_k \right]$$

$$\hat{e}_\theta = \frac{1}{h_2} \left[\frac{\partial x_1}{\partial \theta} \hat{e}_1 + \frac{\partial x_2}{\partial \theta} \hat{e}_2 + \frac{\partial x_3}{\partial \theta} \hat{e}_3 \right]$$

$$= \frac{1}{h_2} \left[-r \sin \theta \hat{e}_1 + r \cos \theta \hat{e}_2 \right]$$

$$= \frac{r}{h_2} \left[-\sin \theta \hat{e}_1 + \cos \theta \hat{e}_2 \right]$$

Requiring $\hat{e}_2 \cdot \hat{e}_2 = 1$

$$\Rightarrow \frac{r^2}{h_2^2} = 1$$

or, $h_2 = r.$

H(w): Show $\hat{e}_3 = \underline{e}_3.$

In cylindrical coordinates

$$\underline{r} = r_i \underline{e}_i$$

$$= u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ u_{\langle 1 \rangle} & u_{\langle 2 \rangle} & u_{\langle 3 \rangle} \end{array}$$

$\sigma_{\langle ij \rangle}$ → angle brackets indicate non-linear coordinates.

Formulas for

$$\left. \begin{array}{l} \nabla \\ \nabla \cdot \\ \nabla \times \end{array} \right\}$$

in curvilinear coords.

∇ in Cartesian:

$$\nabla \cdot = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$$

∇ is **no** curvilinear coords; **indexial notation!!**

$$\nabla f = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial f}{\partial \xi_i}$$

$$\nabla \underline{u} = \sum_i \sum_j \frac{\hat{e}_i}{h_i} \otimes \left\{ \frac{\partial u_{ij}}{\partial \xi_i} \hat{e}_j + u_{ij} \frac{\partial \hat{e}_j}{\partial \xi_i} \right\}$$

$$\nabla \cdot \underline{u} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial \xi_i} \left(\frac{h_1 h_2 h_3}{h_i} u_{\langle i \rangle} \right)$$

$$\nabla \times \underline{u} = \sum_{i,j,k} \frac{\epsilon_{ijk}}{h_j h_k} \frac{\partial}{\partial \xi_j} (u_{\langle k \rangle} h_k) \underline{e}_i$$

Given $\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta + u_z \underline{e}_z$,

we want to find the components of $\underline{\epsilon} = \frac{\nabla \underline{u} + \nabla \underline{u}^T}{2}$

in cylindrical coordinates.

$$\left[\underline{\epsilon} \right] = \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{bmatrix}$$

$$\epsilon_{rr} = ? \quad \epsilon_{r\theta} = ? \quad \dots$$

$$\nabla u \approx \sum_i \sum_j \frac{\hat{e}_i \otimes \hat{e}_j}{h_i} \left(\frac{\partial u_{ij}}{\partial \xi_i} \hat{e}_j + u_{ij} \frac{\partial \hat{e}_j}{\partial \xi_i} \right)$$

$i=1$ terms

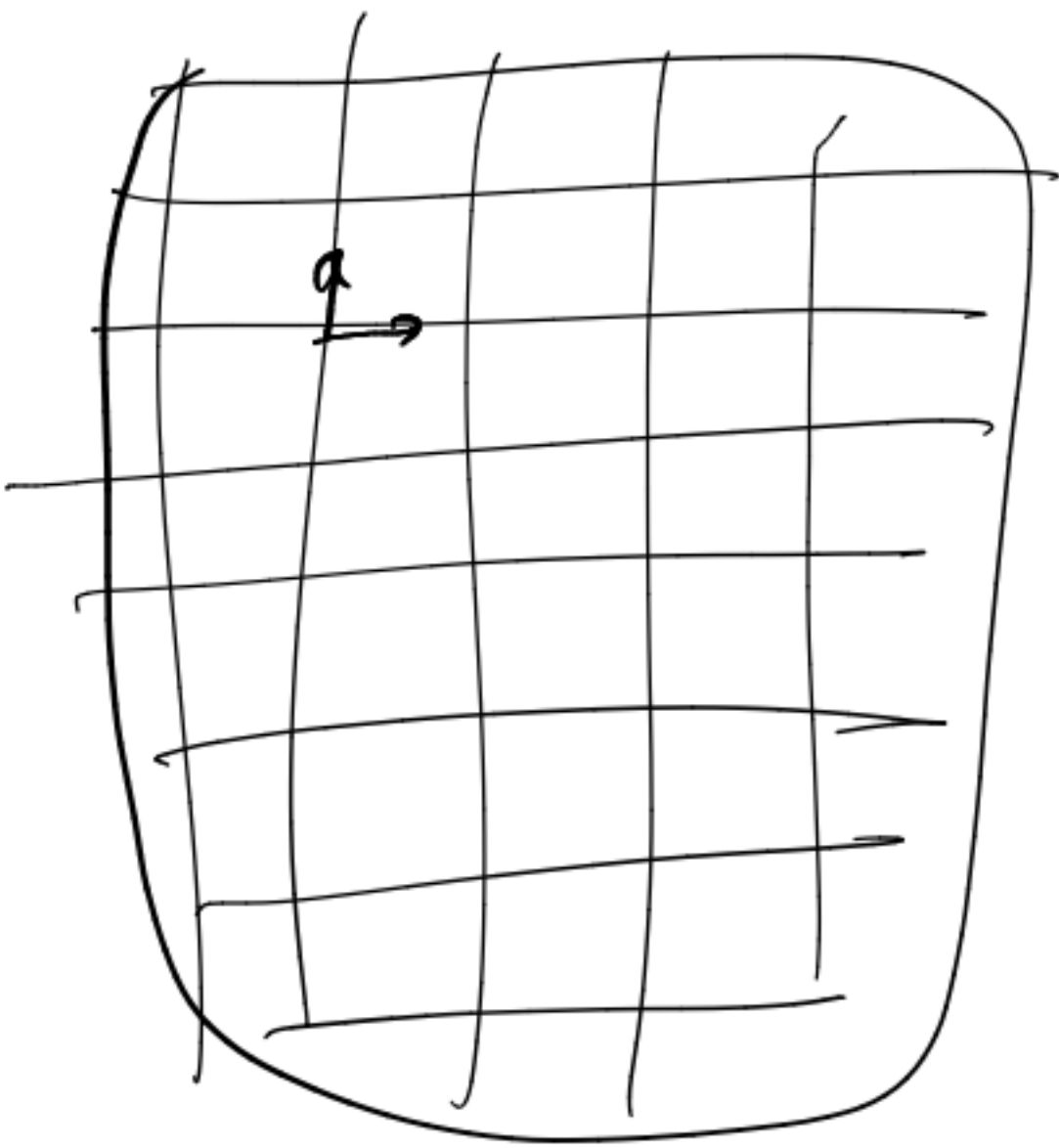
$$= \frac{\hat{e}_r \otimes \hat{e}_r}{1} \left\{ \frac{\partial u_r}{\partial r} \hat{e}_r + u_r \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial u_\theta}{\partial r} \hat{e}_\theta + u_\theta \frac{\partial \hat{e}_\theta}{\partial r} + \frac{\partial u_z}{\partial r} \hat{e}_z + u_z \frac{\partial \hat{e}_z}{\partial r} \right\}$$

$i=2$ terms

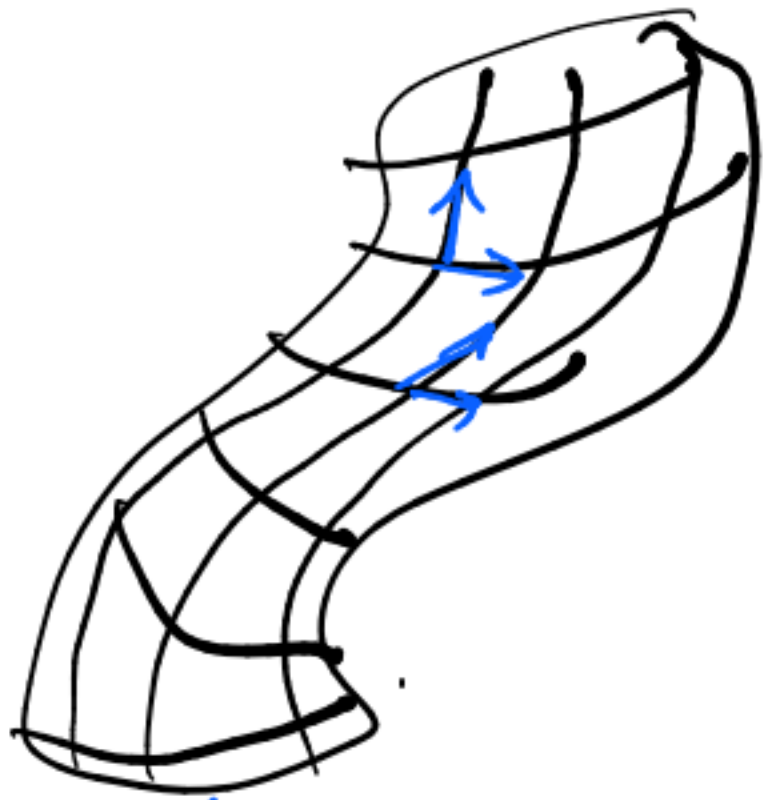
$$\frac{\hat{e}_\theta \otimes \hat{e}_r}{r} \left\{ \frac{\partial u_r}{\partial \theta} \hat{e}_r + u_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta} \hat{e}_\theta + u_\theta \frac{\partial \hat{e}_\theta}{\partial \theta} + \frac{\partial u_z}{\partial \theta} \hat{e}_z + u_z \frac{\partial \hat{e}_z}{\partial \theta} \right\} +$$

$i=3$ terms

$$\frac{\hat{e}_z \otimes \hat{e}_r}{1} \left\{ \frac{\partial u_r}{\partial z} \hat{e}_r + u_r \frac{\partial \hat{e}_r}{\partial z} + \frac{\partial u_\theta}{\partial z} \hat{e}_\theta + u_\theta \frac{\partial \hat{e}_\theta}{\partial z} + \frac{\partial u_z}{\partial z} \hat{e}_z + u_z \frac{\partial \hat{e}_z}{\partial z} \right\}$$



deformation
→



Green &

Zerina → Elasticity
