AS5900

Lecture 15

curvilinear coordinates.

Orthogonal Cartesian	Orthogonal Curvilinear
(x_1, x_2, x_3)	(ξ_1, ξ_2, ξ_3)

$$\xi_i = \xi_i(x_1, x_2, x_3).$$

e-g in cylindrical cordinates:

$$r = \xi_1 = \sqrt{x_1^2 + x_2^2}$$

$$\theta = \xi_2 = \tan^{-1}\frac{x_2}{x_1}$$

$$z = \xi_3 = x_3.$$

$$x_i = x_i(\xi_1, \xi_2, \xi_3)$$

eg in cyl. coordintes

$$x_1 = \cancel{r}^{\,\xi_1} \cos\cancel{\theta}^{\,\xi_2},\quad x_2 = \cancel{r}^{\,\xi_1}\sin\cancel{\theta}^{\,\xi_2}$$

$$x_3 = \cancel{z}.\ \xi_3$$

Unit vectors of the curvilinear coordinate system:

Cartesian u.v.

$$\hat{\underset{\sim}{e}}_1 = \frac{1}{h_1} \frac{\partial x_k}{\partial \xi_1} \underset{\sim}{e}_1$$

curvilinear unit vector along ξ_1

Cartesian

$$\hat{\underset{\sim}{e}}_2 = \frac{1}{h_2} \frac{\partial x_k}{\partial \xi_2} \underset{\sim}{e}_2 \quad \hat{j}$$

$$\hat{\underset{\sim}{e}}_3 = \frac{1}{h_3} \frac{\partial x_k}{\partial \xi_3} \underset{\sim}{e}_3 \quad \hat{\underset{\sim}{k}}$$

h_1, h_2, h_3 — scale factors

— scalar functions.

found by requiring $\hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_1 = 1$

$\hat{\underset{\sim}{e}}_3 \cdot \hat{\underset{\sim}{e}}_3 = 1$; $\hat{\underset{\sim}{e}}_2 \cdot \hat{\underset{\sim}{e}}_2 = 1$

Eg: find $\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3, h_1, h_2, h_3$ for cylindrical coordinates.

$$\hat{\underset{\sim}{e}}_1 = \frac{1}{h_1} \frac{\partial x_k}{\partial \xi_1} \hat{e}_k$$

$$= \frac{1}{h_1}\left[\frac{\partial x_1}{\partial r}\hat{e}_1 + \frac{\partial x_2}{\partial r}\hat{e}_2 + \frac{\partial x_3}{\partial r}\hat{e}_3\right]$$

$$= \frac{1}{h_1}\left[\cos\theta\, \hat{\underset{\sim}{e}}_1 + \sin\theta\, \hat{\underset{\sim}{e}}_2\right]$$

Find h_1 by requiring

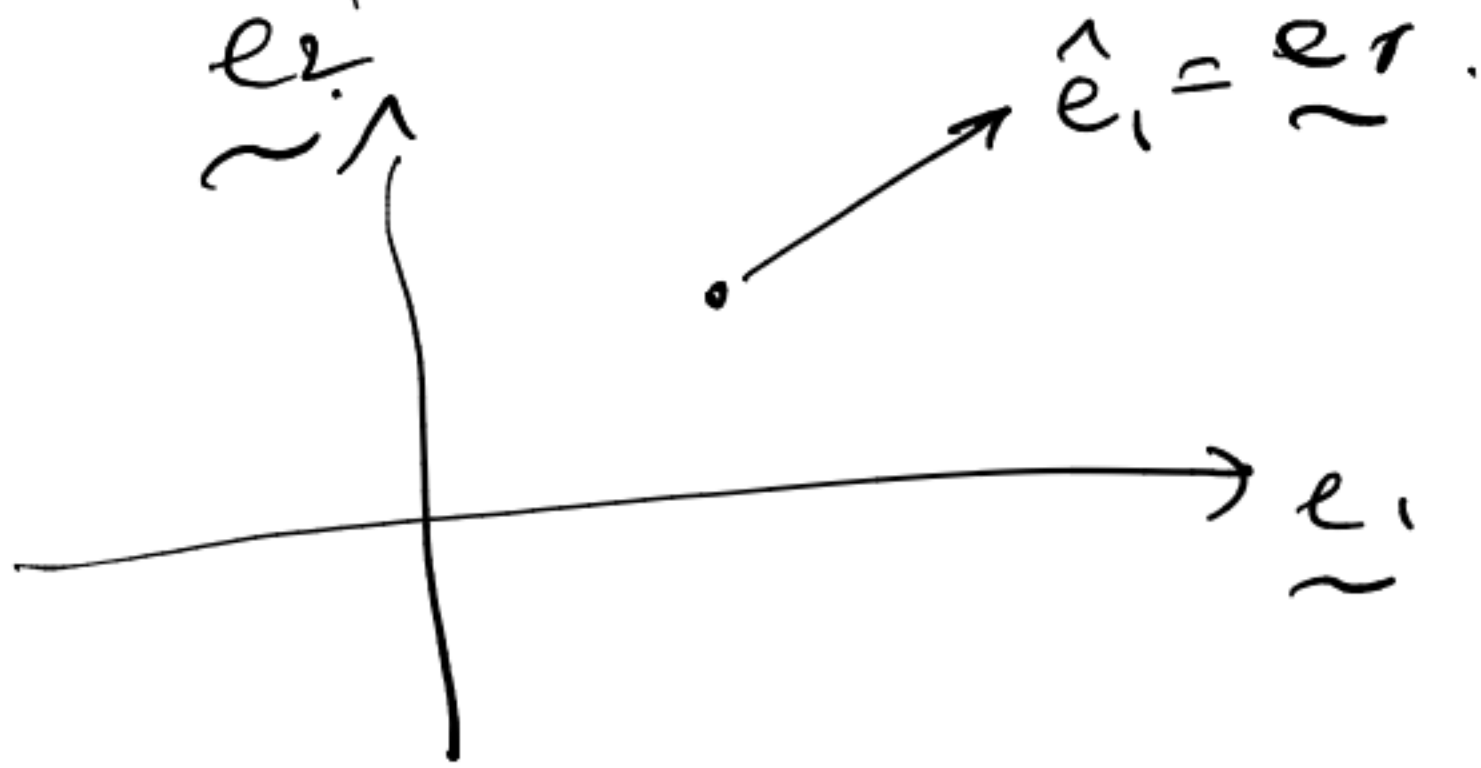
$$\hat{\underset{\sim}{e}}_1 \cdot \hat{\underset{\sim}{e}}_1 = 1$$

$$\frac{1}{h_1^2}\left[\cos^2\theta + \sin^2\theta\right] = 1 \Rightarrow h_1 = 1.$$

h_i always +ve.

$$\hat{\underset{\sim}{e}}_r = \cos\theta \, \underset{\sim}{e}_1 + \sin\theta \, \underset{\sim}{e}_2 .$$

$$\hat{\underset{\sim}{e}}_r = \hat{\underset{\sim}{e}}_1$$



$$\hat{\underset{\sim}{e}}_2 = \frac{1}{h_2}\left[\frac{\partial x_k}{\partial \theta}\underset{\sim}{e}_k\right]$$

$$\hat{\underset{\sim}{e}}_\theta = \frac{1}{h_2}\left[\frac{\partial x_1}{\partial \theta}\underset{\sim}{e}_1 + \frac{\partial x_2}{\partial \theta}\underset{\sim}{e}_2 + \frac{\partial x_3}{\partial \theta}\underset{\sim}{e}_3\right]$$

$$= \frac{1}{h_2}\left[-r\sin\theta \, \underset{\sim}{e}_1 + r\cos\theta \, \underset{\sim}{e}_2\right]$$

$$= \frac{r}{h_2}\left[-\sin\theta \, \underset{\sim}{e}_1 + \cos\theta \, \underset{\sim}{e}_2\right]$$

Requiring $\underset{\sim}{\hat{e}_2} \cdot \underset{\sim}{\hat{e}_2} = 1$

$$\Rightarrow \quad \frac{r^2}{h_2^2} = 1$$

or, $h_2 = r$.

H/W: Show $\underset{\sim}{\hat{e}_3} = \underset{\sim}{e_3}$.

In cylindrical coords

$$\underset{\sim}{u} \neq u_i \underset{\sim}{e_i}$$

$$\begin{array}{ccccc} = u_r \underset{\sim}{\hat{e}_r} & + & u_\theta \underset{\sim}{\hat{e}_\theta} & + & u_z \underset{\sim}{\hat{e}_z} \\ \downarrow & & \downarrow & & \downarrow \\ u_{\langle 1 \rangle} & & u_{\langle 2 \rangle} & & u_{\langle 3 \rangle}. \end{array}$$

$\sigma_{\langle ij \rangle} \rightarrow$ angle brackets indicate curvilinear coordinates.

Formulas for

$\left.\begin{array}{l}\nabla \\ \nabla\cdot \\ \nabla\times\end{array}\right\}$ in curvilinear coords.

∇ in Cartesian:

$$\nabla\cdot = \underset{\sim}{e}_i \frac{\partial\,\cdot}{\partial x_i}$$

∇ is Curvilinear Coords:

No indicial notation!!

$$\nabla f = \sum_i \hat{\underset{\sim}{e}}_i \frac{1}{h_i} \frac{\partial f}{\partial \xi_i}$$

$$\nabla \underset{\sim}{u} = \sum_i \sum_j \frac{\hat{\underset{\sim}{e}}_i}{h_i} \otimes \left\{ \frac{\partial u_{\langle j\rangle}}{\partial \xi_i} \hat{\underset{\sim}{e}}_j + u_{\langle j\rangle} \frac{\partial \hat{\underset{\sim}{e}}_j}{\partial \xi_i} \right\}$$

$$\nabla \cdot \underset{\sim}{u} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial \xi_i} \left(\frac{h_1 h_2 h_3}{h_i} u_{\langle i \rangle} \right),$$

$$\nabla \times \underset{\sim}{u} = \sum_i \sum_j \sum_k \frac{\varepsilon_{ijk}}{h_j h_k} \frac{\partial}{\partial \xi_j} \left(u_{\langle k \rangle} h_k \right) \hat{\underset{\sim}{e}}_i .$$

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given $\underset{\sim}{u} = u_r \underset{\sim}{e_r} + u_\theta \underset{\sim}{e_\theta} + u_z \underset{\sim}{e_z}$, we want to find the components of $\underset{\sim}{\varepsilon} = \dfrac{\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T}{2}$

in cylindrical coordinates.

$$\left[\underset{\sim}{\varepsilon} \right] = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{zr} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{bmatrix}.$$

$\varepsilon_{rr} = ?$ $\varepsilon_{r\theta} = ?$ ....

$$\nabla \underset{\sim}{u} = \sum_i \sum_j \frac{\hat{e}_i \otimes}{h_i}\left(\frac{\partial u_{\langle j\rangle}}{\partial \xi_i}\underset{\sim}{\hat{e}_j} + u_{\langle j\rangle}\frac{\partial \hat{e}_j}{\partial \xi_i}\right)$$

$$= \frac{\underset{\sim}{\hat{e}_r}}{1}\otimes\left\{\frac{\partial u_r}{\partial r}\underset{\sim}{\hat{e}_r} + u_r \cancel{\frac{\partial \underset{\sim}{\hat{e}_r}}{\partial r}}^{\,0\text{ p.11.}} + \frac{\partial u_\theta}{\partial r}\underset{\sim}{\hat{e}_\theta} + u_\theta \cancel{\frac{\partial \hat{e}_\theta}{\partial r}}^{\,0\text{ p.11}} + \frac{\partial u_z}{\partial r}\underset{\sim}{\hat{e}_z} + u_z \cancel{\frac{\partial \hat{e}_z}{\partial r}}^{\,0}\right\}$$

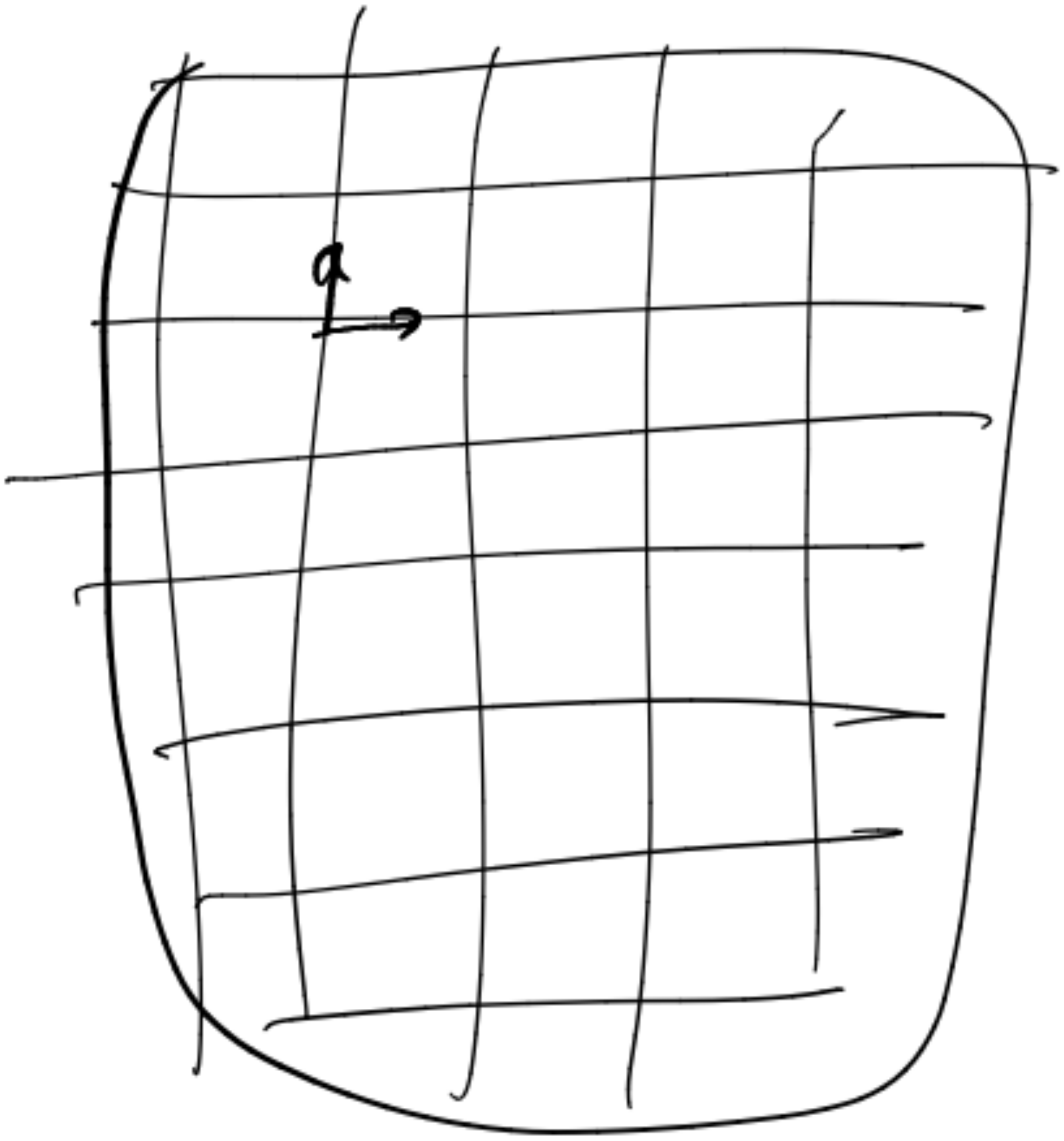
$i=1$ terms

$$+ \frac{\hat{e}_\theta}{r}\otimes\left\{\frac{\partial u_r}{\partial \theta}\underset{\sim}{\hat{e}_r} + u_r \cancel{\frac{\partial \underset{\sim}{\hat{e}_r}}{\partial \theta}}^{\,\hat{e}_\theta} + \frac{\partial u_\theta}{\partial \theta}\underset{\sim}{\hat{e}_\theta} + u_\theta \cancel{\frac{\partial \hat{e}_\theta}{\partial \theta}}^{\,-\underset{\sim}{\hat{e}_r}} + \frac{\partial u_z}{\partial \theta}\underset{\sim}{\hat{e}_z} + u_z \cancel{\frac{\partial \underset{\sim}{\hat{e}_z}}{\partial \theta}}^{\,0}\right\} +$$

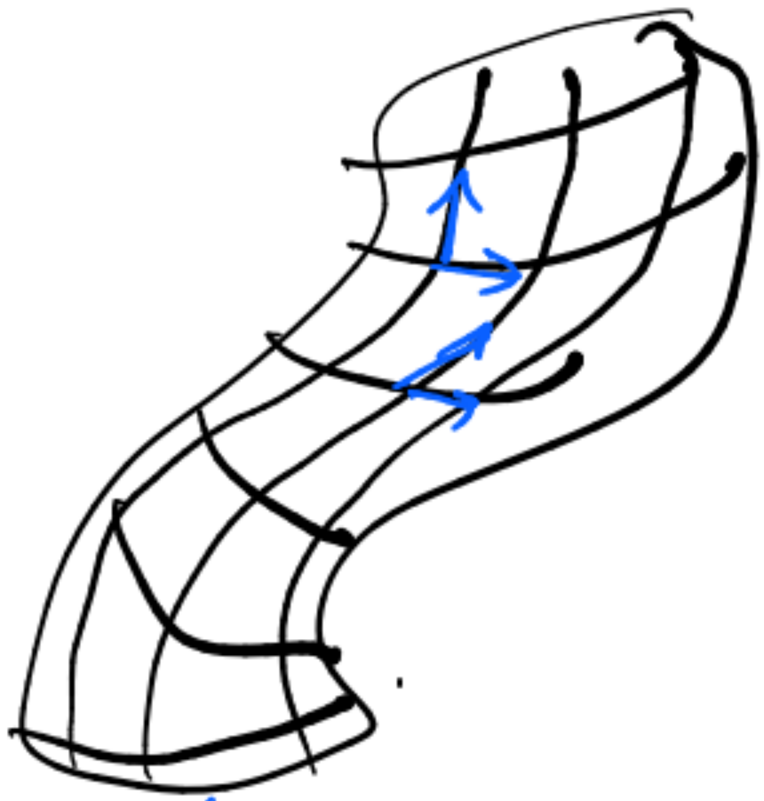
$i=2$ terms

$i=3$ terms

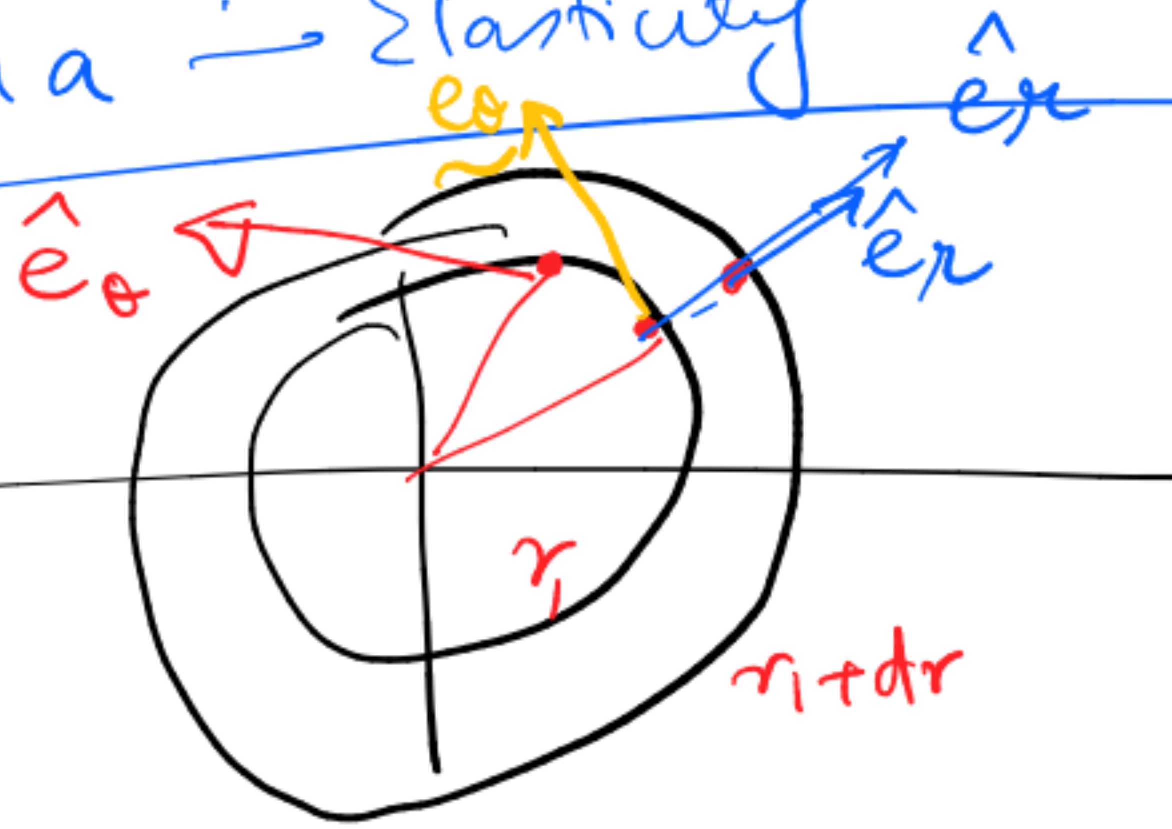
$$\frac{\hat{e}_z}{1}\otimes\left\{\frac{\partial u_r}{\partial z}\hat{e}_r + u_r \cancel{\frac{\partial \underset{\sim}{\hat{e}_r}}{\partial z}}^{\,0} + \frac{\partial u_\theta}{\partial z}\hat{e}_\theta + u_\theta \cancel{\frac{\partial \hat{e}_\theta}{\partial z}}^{\,0} + \frac{\partial u_z}{\partial z}\hat{e}_z + u_z \cancel{\frac{\partial \underset{\sim}{\hat{e}_z}}{\partial z}}^{\,0}\right\}$$



deform.



Green &
Zerna $\longrightarrow$ Elasticity



$$\left.\frac{\partial \hat{e}_r}{\partial r}\right|_{\theta, z \text{ fixed}} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta}$$

AS5900

Lecture 16.

!! No index notation in curvilinear coordinates !!

(1.9.10): (No derivation)

(a) $$\frac{\partial \hat{\underset{\sim}{e}}_m}{\partial \xi_m} = -\frac{1}{h_n}\frac{\partial h_m}{\partial \xi_n}\hat{e}_n - \frac{1}{h_r}\frac{\partial h_m}{\partial \xi_r}\hat{e}_r$$

if $m \neq n \neq r$.

(b) $$\frac{\partial \hat{e}_m}{\partial \xi_n} = \frac{1}{h_m}\frac{\partial h_n}{\partial \xi_m}\hat{e}_n$$

if $m \neq n$.

Apply this to find a few of the derivatives in page 8

$$\left.\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= 1. \end{aligned}\right\} \text{ in cyl. coords.}$$

$\xi_1 = r$
$\xi_2 = \theta$
$\xi_3 = z.$

p.10 $n = 2;\ r = 3$

$$\frac{\partial \hat{e}_r}{\partial r} = -\frac{1}{r}\frac{\partial(1)}{\partial \theta}\hat{\underset{\sim}{e}}_\theta - \frac{1}{1}\frac{\partial(1)}{\partial z}\hat{e}_z$$

$$= 0.$$

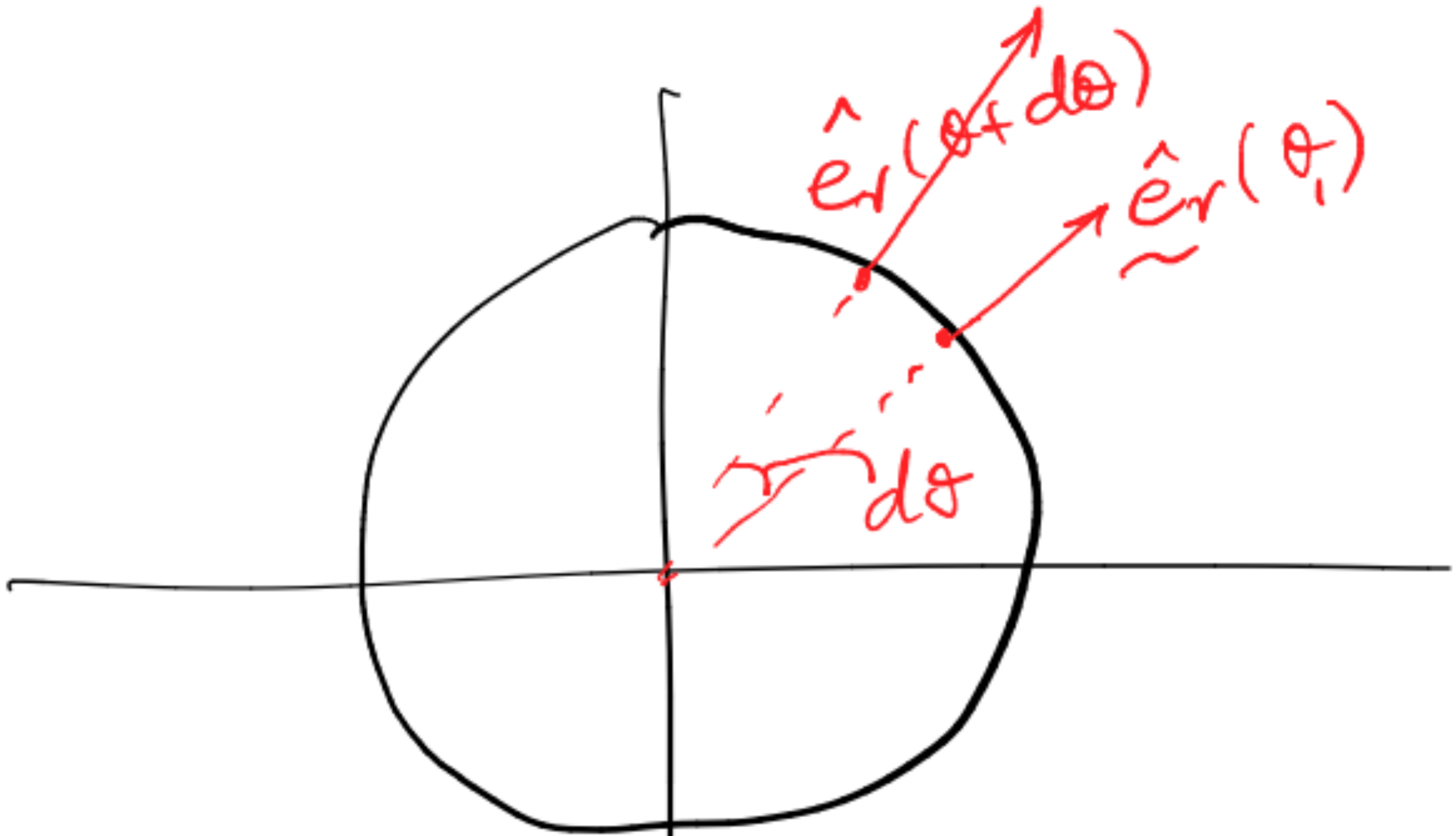
$m = 1$

$$\frac{\partial \hat{\underset{\sim}{e}}_\theta}{\partial r} = \frac{1}{r}\frac{\partial(1)}{\partial \theta}\hat{\underset{\sim}{e}}_r = \underset{\sim}{0}.$$

$m = 2$
$n = 1$

$m = 1,\ n = 2$

$$\frac{\partial \hat{\underset{\sim}{e}}_r}{\partial \theta} = \frac{1}{1}\cdot\frac{\partial(r)}{\partial r}\hat{\underset{\sim}{e}}_\theta = \hat{\underset{\sim}{e}}_\theta.$$



$m=2; \quad n=1; \quad r=3$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} \overset{ⓐ}{=} -\frac{1}{1}\frac{\partial r}{\partial r}\hat{\underset{\sim}{e}}_r - \cancelto{0}{\frac{1}{1}\frac{\partial r}{\partial z}}\hat{\underset{\sim}{e}}_z$$

$$= -\hat{e}_r$$

In the cartesian world if we have

$$[\underset{\sim}{\varepsilon}]_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

$$\underset{\sim}{\varepsilon} = \varepsilon_{11}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_1} + \varepsilon_{12}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_2} + \varepsilon_{13}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_3}$$
$$\cdots\cdots + \varepsilon_{33}\, \underset{\sim}{e_3} \otimes \underset{\sim}{e_3}$$

In the cylindrical coord. system,

$$\underset{\sim}{\varepsilon} = \varepsilon_{rr}\, \underset{\sim}{e_r} \otimes \underset{\sim}{e_r} + \varepsilon_{r\theta}\, \underset{\sim}{e_r} \otimes e_\theta + \varepsilon_{rz}\, \underset{\sim}{e_r} \otimes e_z + \varepsilon_{\theta r}\, \underset{\sim}{e_\theta} \otimes \underset{\sim}{e_r} + \cdots + \varepsilon_{zz}\, \underset{\sim}{e_z} \otimes \underset{\sim}{e_z}$$

$$\left.\begin{aligned}\nabla \underset{\sim}{u} = \; & (\nabla \underset{\sim}{u})_{rr}\, \underset{\sim}{e_r} \otimes \underset{\sim}{e_r} + \\ & (\nabla \underset{\sim}{u})_{r\theta}\, \underset{\sim}{e_r} \otimes \underset{\sim}{e_\theta} + \\ & (\nabla u)_{rz}\, \underset{\sim}{e_r} \otimes \underset{\sim}{e_z} + \\ & \ldots \\ & (\nabla u)_{zz}\, \underset{\sim}{e_z} \otimes \underset{\sim}{e_z} .\end{aligned}\right\} \quad (2)$$

Compare (2) w/ that on page (8):

$$(\nabla u)_{rr} = \frac{\partial u_r}{\partial r}.$$

$$(\nabla u)_{r\theta} = \frac{\partial u_\theta}{\partial r}$$

$$(\nabla u)_{rz} = \frac{\partial u_z}{\partial r}.$$

$$(\nabla u)_{\theta r} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}.$$

$$(\nabla u)_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$(\nabla u)_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

$\underline{\text{IIly}}$: $(\nabla u)_{zr}$; $(\nabla u)_{z\theta}$; $(\nabla u)_{zz}$.

$\varepsilon_{rr} = (\nabla u)_{rr}$

$\varepsilon_{\theta\theta} = (\nabla u)_{\theta\theta}$

$\varepsilon_{zz} = (\nabla u)_{zz}$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{(\nabla u)_{r\theta} + (\nabla u)_{\theta r}}{2}$$

$$= \frac{1}{2}\left\{\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\right\}.$$

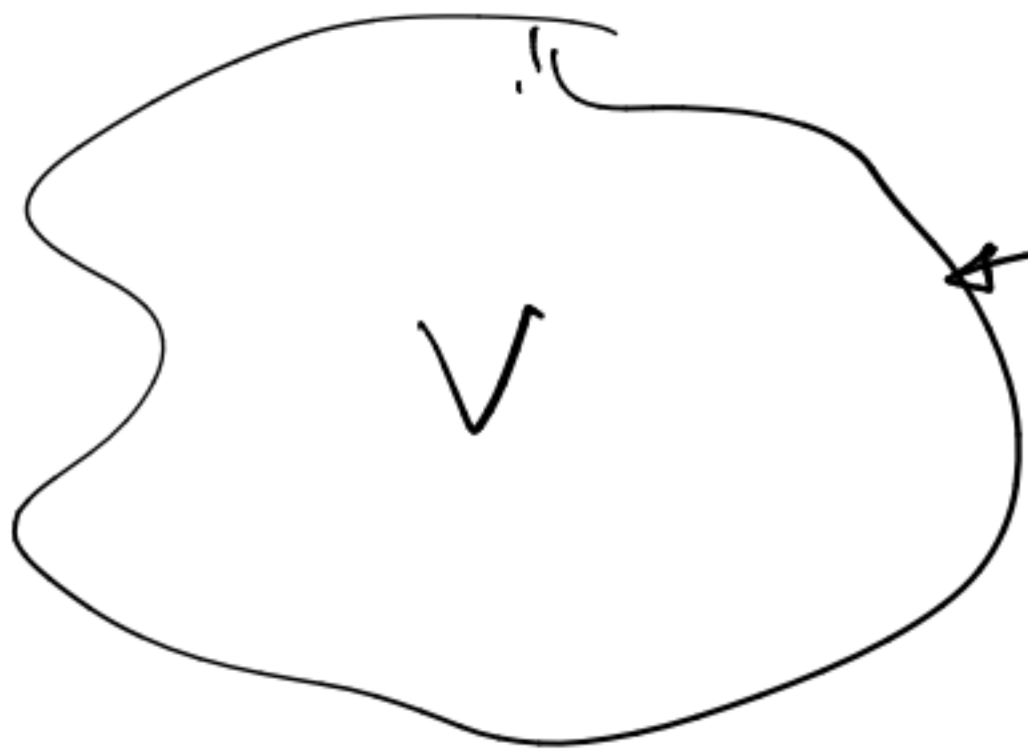
ε_{rz}, $\varepsilon_{\theta z}$ — calculate similarly.

H/W: everything behind Chap 2

except Ex. 2.18

Chap 3

Forces, tractions & stresses.

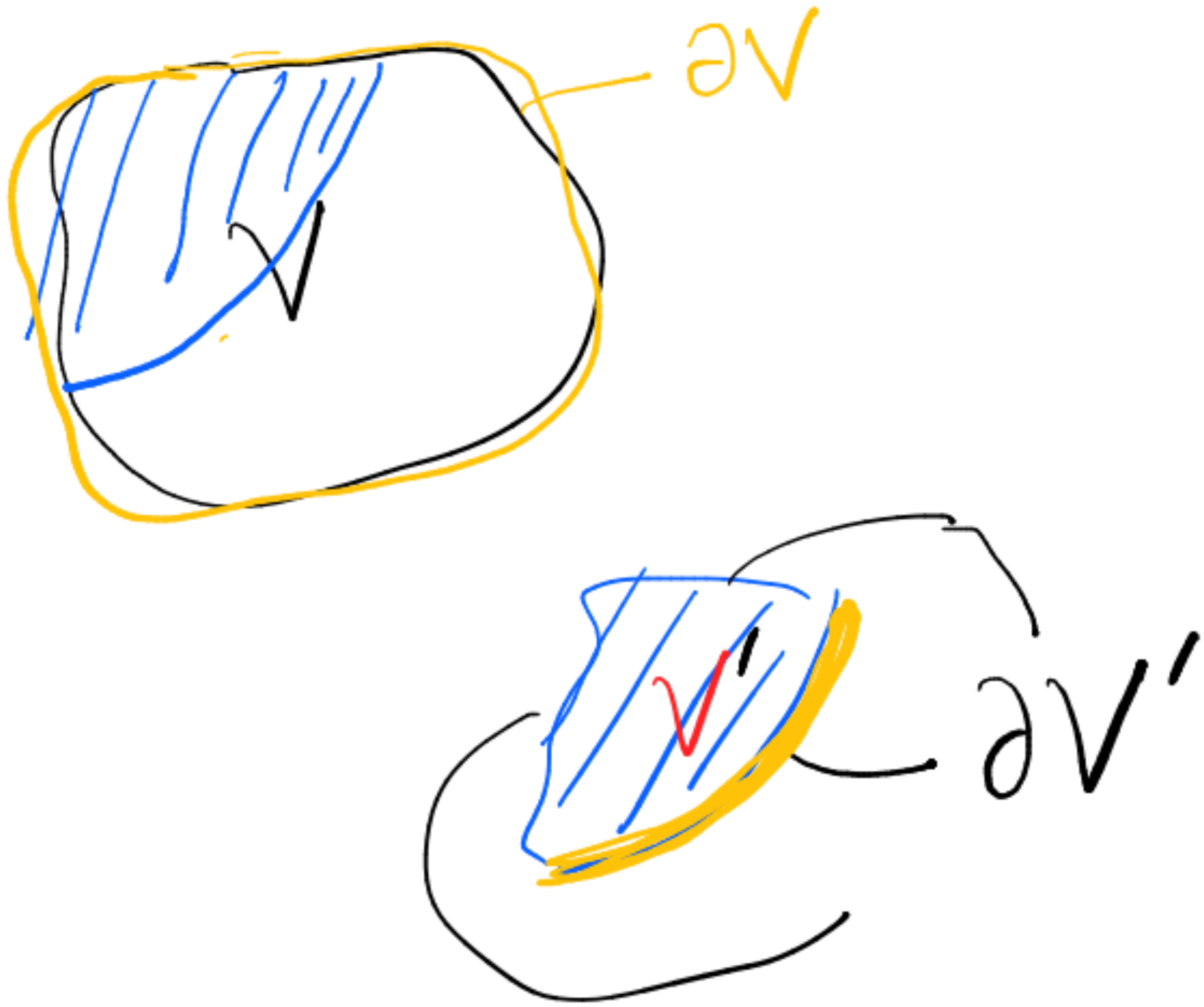


∂V = surface of V.

Two types for forces can be applied to the body.

(1) Body force — fields that produce action @ a distance. "No touch forces".

(2) Surface force — force produced by contact.



∂V
V
V'
∂V'