AS 5900

Lecture - 2

Rule 1 : Sum over repeated indices.

$$\underset{\sim}{u} = u_1 \underset{\sim}{e}_1 + u_2 \underset{\sim}{e}_2 + u_3 \underset{\sim}{e}_3$$

$u_1 + u_2 + u_3$ — Cannot be shortened using Index notation.

$$|\underset{\sim}{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u_i u_i}$$

↑ using Rule 1.

Rule 2: $\dfrac{\partial \odot}{\partial x_j}$ is written as $\odot_{,j}$

$$\frac{\partial T}{\partial x_2} = T_{,2}$$

$$\frac{\partial u_i}{\partial x_j} = u_{i,j}$$

etc.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2}$$

$$= \sum_{i=1}^{3} \frac{\partial^2 T}{\partial x_i^2} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\right)$$

Rules 1, 2

$$\equiv T_{,ii}$$ → Index form of $\nabla^2 T$

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2 extra symbols δ_{ij} & ε_{ijk}.

δ_{ij} = Kronecker delta Symbol

ε_{ijk} = Levi - Civita Symbol.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

$$\underset{\sim}{u} = u_1 \underset{\sim}{e_1} + u_2 \underset{\sim}{e_2} + u_3 \underset{\sim}{e_3}$$

$$\underset{\sim}{e_1} \to \underset{\sim}{i}, \quad \underset{\sim}{e_2} \to \underset{\sim}{j}, \quad \underset{\sim}{e_3} \to \underset{\sim}{k}$$

$$\delta_{ij}\, u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3$$

Suppose $i=1$. $= u_1$

" $i=2$ $= u_2$

" $i=3$ $= u_3$.

$$\boxed{\delta_{ij}\, u_j = u_i}$$ "Contraction"

Essentially multiplying by δ_{ij} replaces repeated index (j) with free index. (i)

$$\delta_{ij}\, \varepsilon_{ij} = \text{"}\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij}\, \varepsilon_{ij}\text{"}$$

(9 terms).

$$\delta_{ij} \cdot \varepsilon_{ij} = \varepsilon_{ii}. = \sum_{i=1}^{3} \varepsilon_{ii}$$

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$= \text{Trace}\,(\varepsilon).$$

Check if this is actually true.

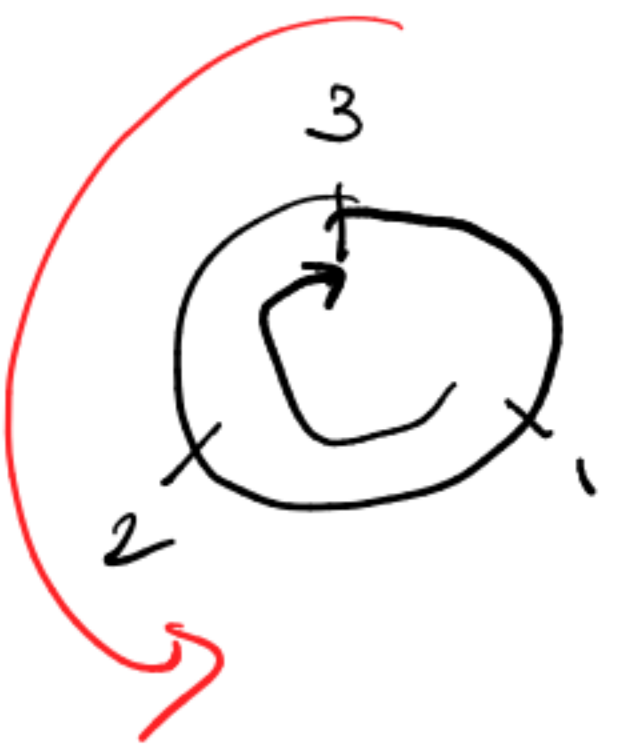
$$\underset{1}{\cancel{\delta_{11}}}\,\varepsilon_{11} + \underset{1}{\cancel{\delta_{22}}}\,\varepsilon_{22} + \underset{1}{\cancel{\delta_{33}}}\,\varepsilon_{33} +$$

$$\underset{0}{\cancel{\delta_{12}}}\,\varepsilon_{12} + \underset{0}{\cancel{\delta_{13}}}\,\varepsilon_{13} + \underset{0}{\cancel{\delta_{23}}}\,\varepsilon_{23} +$$

$$\underset{0}{\cancel{\delta_{21}}}\,\varepsilon_{21} + \underset{0}{\cancel{\delta_{31}}}\,\varepsilon_{31} + \underset{0}{\cancel{\delta_{32}}}\,\varepsilon_{32}$$

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1) \text{ or } (2,1,3) \text{ or } (1,3,2) \\ 0 & \text{otherwise} \end{cases}$$



$$\varepsilon_{111} = \varepsilon_{112} = \varepsilon_{212} = 0$$

etc.

Use of ε_{ijk}. Let

$$\underset{\sim}{u} = u_i \underset{\sim}{e_i} \; ; \quad \underset{\sim}{v} = v_i \underset{\sim}{e_i}$$

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{e_1} & \underset{\sim}{e_2} & \underset{\sim}{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\parallel$$

$$\underset{\sim}{w}$$

$$\underset{\sim}{w} = \underset{\sim}{e_1}(u_2 v_3 - v_2 u_3) - \underset{\sim}{e_2}(u_1 v_3 - u_3 v_1) + \underset{\sim}{e_3}(u_1 v_2 - u_2 v_1). \qquad (1)$$

In indicial notation:

$$w_i = \varepsilon_{ijk} \, u_j v_k \qquad (2)$$

We should check (1) & (2) are the same.

$$w_1 = \varepsilon_{\underline{1}jk} \, u_j v_k$$

$$\begin{aligned} &= \varepsilon_{123}^{+1} \, u_2 v_3 + && (j=2,\ k=3) \\ &\quad \varepsilon_{132}^{-1} \, u_3 v_2 + && (j=3,\ k=2) \\ &\quad \varepsilon_{122} \, u_2 v_2 + && (j=2,\ k=2) \\ 0 \quad &\quad \varepsilon_{133} \, u_3 v_3 + && (j=3,\ k=3) \end{aligned}$$

Similarly show ω_2 & ω_3 expressions also match.

$$\nabla T = \frac{\partial T}{\partial x_1}\,\underset{\sim}{e_1} + \frac{\partial T}{\partial x_2}\,\underset{\sim}{e_2} + \frac{\partial T}{\partial x_3}\,\underset{\sim}{e_3}.$$

Index Notation

$$\overset{\downarrow}{=} \quad T_{,i}\,\underset{\sim}{e_i}.$$

Rules 1, 2

$$\nabla \underset{\sim}{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$u_1\,\underset{\sim}{e_1} + u_2\,\underset{\sim}{e_2} + u_3\,\underset{\sim}{e_3}.$$

Dyadic Product:

$$\underset{\sim}{e_1} \cdot \underset{\sim}{e_1} = 1$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_i v_i \text{ — scalar.}$$

$$\underset{\sim}{u} = u_1 \underset{\sim}{e_1} + u_2 \underset{\sim}{e_2} + u_3 \underset{\sim}{e_3}.$$

$$\underset{\sim}{v} = v_1 \underset{\sim}{e_1} + v_2 \underset{\sim}{e_2} + v_3 \underset{\sim}{e_3}.$$

Dot product

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Dyadic product

$$\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}}_{1\times 3} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix} = \underset{\sim}{u} \otimes \underset{\sim}{v}.$$

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}_{1\times 3} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{3\times 1} = \begin{pmatrix} u_1 v_1 + u_2 v_2 + u_3 v_3 \end{pmatrix}_{1\times 1}$$

$$\underset{\sim}{e_1} \otimes \underset{\sim}{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{\sim}{e_1} \otimes \underset{\sim}{e_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{ij}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}.$$

(Index notation)

RHS =

$$a_{11}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_1} + a_{12}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_2} + \cdots$$

$$= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots \text{(7 more terms)}$$

Revisit ☆ on p. 6:

$$\boxed{\nabla \underset{\sim}{u} = u_{i,j}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}.}$$

$$\nabla \cdot \underset{\sim}{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= u_{i,i} \text{ (Index notation)}$$

$$\nabla \times \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{e_1} & \underset{\sim}{e_2} & \underset{\sim}{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$(\nabla \times \underset{\sim}{u})_i \overset{\text{Rule 1}}{=} \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k$$

$$\boxed{(\nabla \times \underset{\sim}{u})_i \overset{\text{Rule 2}}{=} \varepsilon_{ijk}\, u_{k,j}.}$$

Two relations between δ and ε:

$$\varepsilon_{ijk}\, \varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}.$$

H/W: Verify for $ijk = (1,2,3)$
$pqr = (2,3,1)$

H/W: Obtain an expression for

$\varepsilon_{ijk}\, \varepsilon_{iqr} = \underline{\qquad\qquad}$ (in terms of δ)

H/W: Calculate δ_{ii}:

H/W: Show: $\det(a_{ij}) = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} a_{ip} a_{jq} a_{kr}$

$\S 1.3$