A 55900

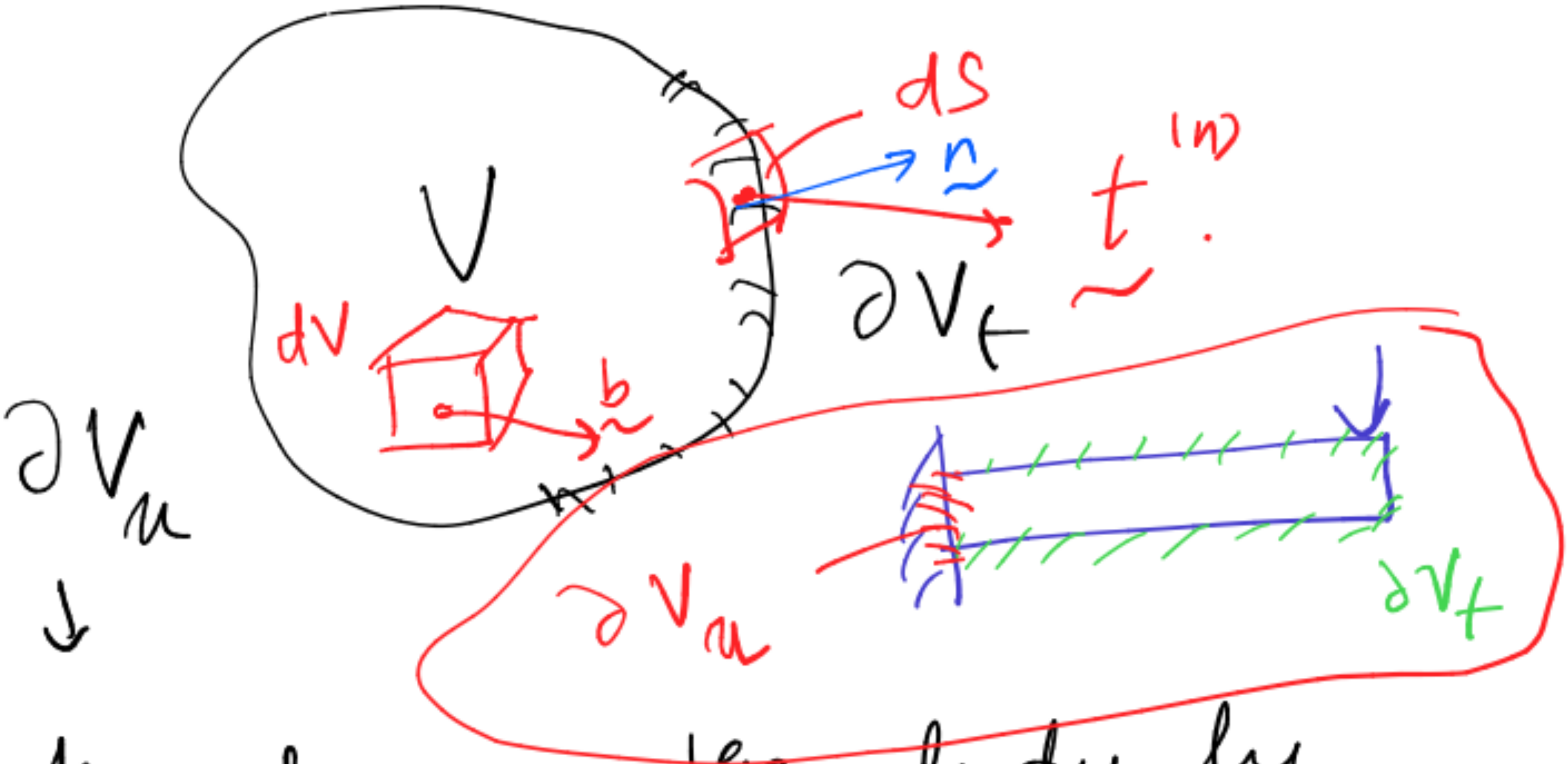
Lecture 28

Strain energy principle

Def: Elastic body: Work done on the body is stored as elastic strain energy in the body.

A Hookean linear elastic solid is only one type of elastic body.

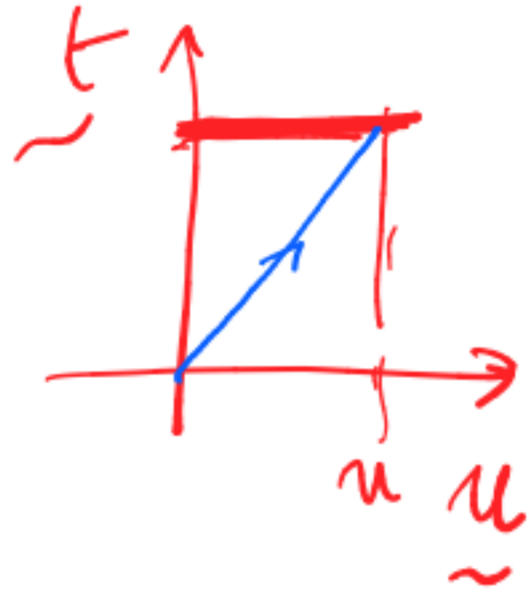
Deriving an expression for strain energy.



Work done on the body by surface tractions & body forces:

$$\iint_{\partial V_t} \left(\underset{\sim}{t}^{(n)} dS\right)\cdot \underset{\sim}{u} + \iiint_V \left(\underset{\sim}{b}\, dV\right)\cdot \underset{\sim}{u}.$$

applied surface traction.



$$\underset{\sim}{t}^{(n)} = \underset{\sim}{\sigma}\,\underset{\sim}{n}.$$

$$\sigma_{ij} n_j$$

$$= \iint_{\partial V_t} t_i^{(n)} u_i \, dS + \iiint_V b_i u_i \, dV.$$

$$= \iint_{\partial V_t} \underbrace{(\sigma_{ij} u_i)}_{p_j} n_j \, dS + \iiint_V b_i u_i \, dV$$

$$\iint_{\partial V_t} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS$$

Over ∂V_u, $p_j = \sigma_{ij} u_i = \sigma_{ij}(0) = 0$.

$$\therefore \iint_{\partial V} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS = \iint_{\partial V_t} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS + \iint_{\partial V_u} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS$$

$$\therefore \iint_{\partial V} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS = \iint_{\partial V_t} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS.$$

$$= \iint_{\partial V} \underset{\sim}{p} \cdot \underset{\sim}{n} \, dS + \iiint_V b_i u_i \, dV.$$

Apply divergence thm. to the 1st term:

$$\iiint_V \nabla \cdot \underset{\sim}{p} \, dV + \iiint_V b_i u_i \, dV$$

$$= \iiint_V p_{j,j} \, dV + \iiint_V b_i u_i \, dV$$

$$= \iiint_V \left\{ (\sigma_{ij} u_i)_{,j} + b_i u_i \right\} dV$$

$$= \iiint_V \left\{ \underbrace{\sigma_{ij,j} u_i + \sigma_{ij} u_{i,j} + b_i}_{= 0 \text{ by eqbm.}} u_i \right\} dV.$$

$$= \iiint_V \sigma_{ij} \left(\varepsilon_{ij} + \omega_{ij} \right) dV.$$

ε_{ij}: Symm $\varepsilon_{ij} = \varepsilon_{ji}$

ω_{ij}: skew $\omega_{ij} = -\omega_{ji}$.

$$\sigma_{ij}\,\omega_{ij} = 0$$

why? $\sigma_{ij} = \sigma_{ji}$

$\omega_{ij} = -\omega_{ji}$

$$\sigma_{ij}\,\omega_{ij} \underset{i \leftrightarrow j}{=} \sigma_{ji}\,\omega_{ji}$$

$$= \sigma_{ij}(-\omega_{ij})$$

$$= -\sigma_{ij}\,\omega_{ij}$$

Compare

$$\Rightarrow \sigma_{ij}\,\omega_{ij} = 0.$$

Work done on the body by surface tractions & body forces =

$$\iiint_V \underbrace{(\sigma_{ij}\,\varepsilon_{ij})}\, dV.$$

$$\mathcal{U}'$$

Work done on the body =

$$\iiint_V \mathcal{U}' \, dV$$

strain energy density.

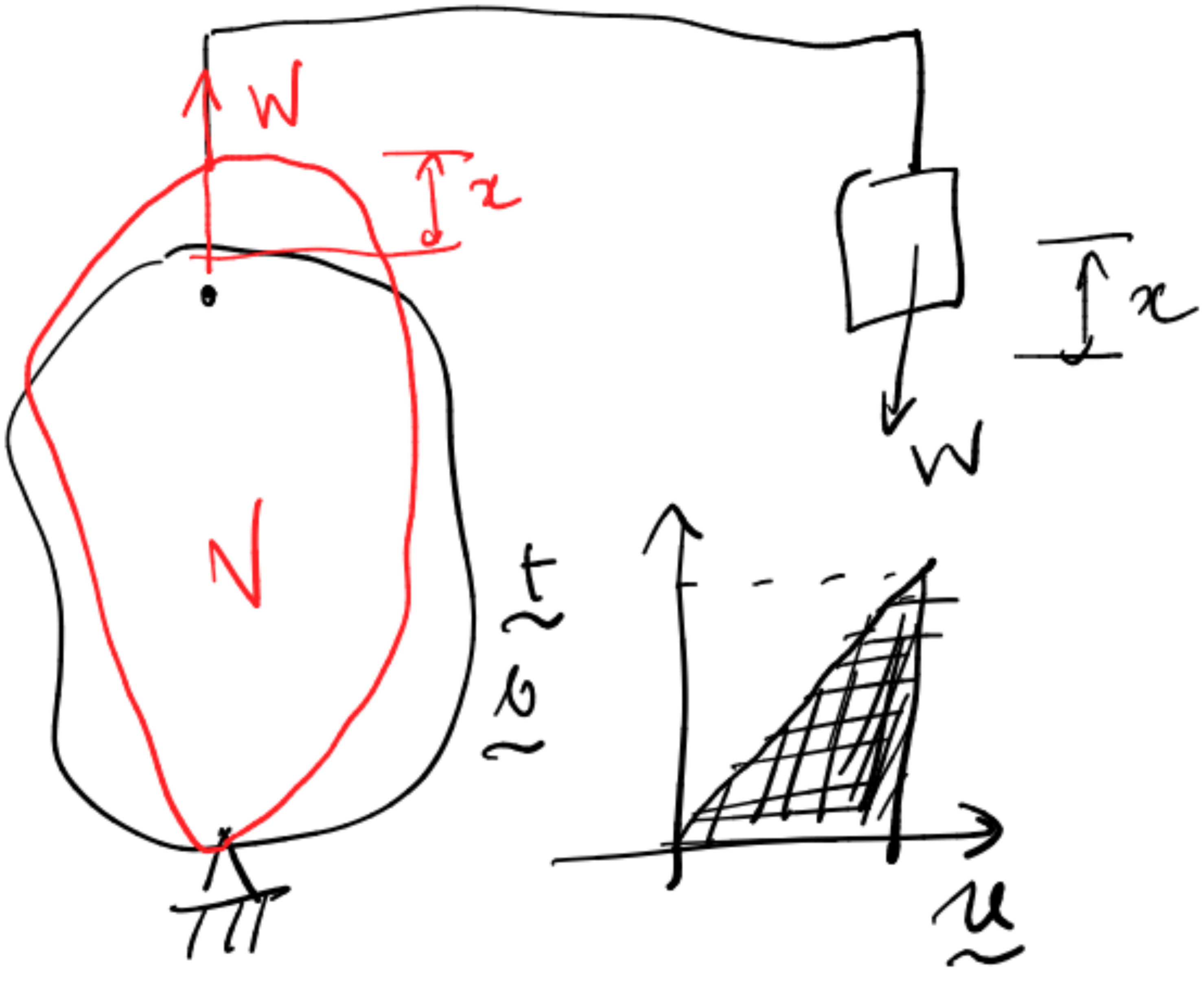
True for all elastic body, not only for linear elastic bodies. loaded instantaneously (no inertia effects)

Now consider linear elastic bodies.

wherein

$\underset{\sim}{t} \propto \underset{\sim}{u}$.

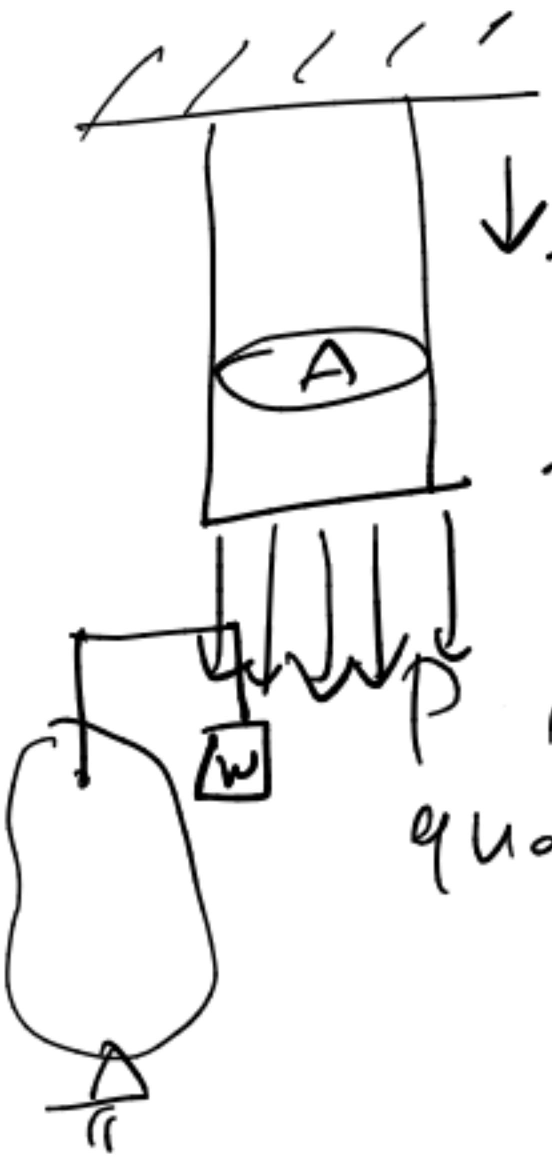
$\underset{\sim}{b} \propto \underset{\sim}{u}$.



If this body were loaded quasistatically, the work done = strain energy $= \frac{1}{2}\iiint u' \, dV$.

We define strain energy density for a linear elastic body as

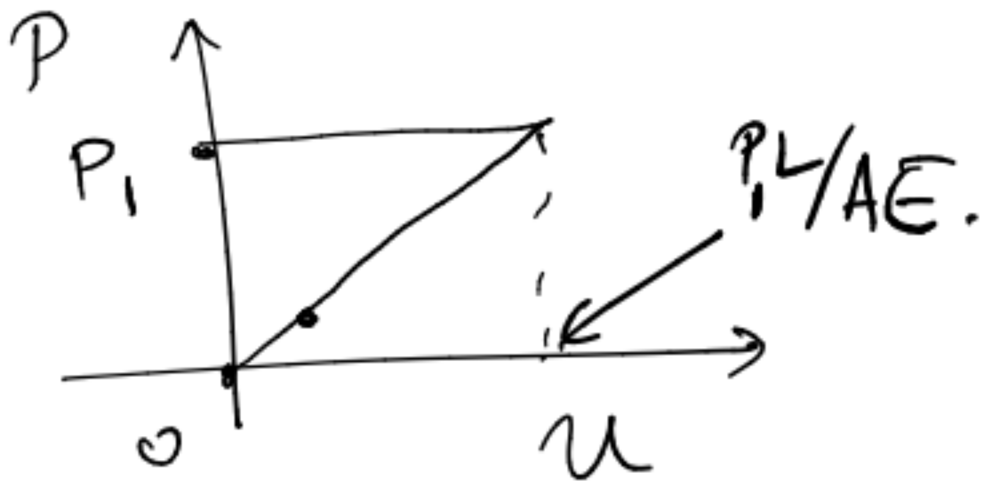
$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}.$$



$\underset{\sim}{i}$

$\underset{\sim}{t}^{(n)} = \frac{P}{A}\underset{\sim}{i}$

P increases quasistatically from 0 to P_1



$\mathcal{U}$ for linear elastic solids

$$\mathcal{U} = \frac{1}{2}\sigma_{ij}\, e_{ij}$$

linear elastic

$$\stackrel{\downarrow}{=} \frac{1}{2}\left(\lambda e_{kk}\delta_{ij} + 2\mu e_{ij}\right) e_{ij}$$

$$= \frac{1}{2}\left(\lambda e_{ii}\, e_{jj} + 2\mu\, e_{ij}\, e_{ij}\right).$$

Non-indicial form:

$$= \frac{1}{2}\left(\lambda\left(e_{11} + e_{22} + e_{33}\right)^2 + 2\mu\left(e_{11}e_{11} + e_{12}e_{12} + \ldots + e_{33}e_{33}\right)\right)$$

Every term is non-negative

So, $\mathcal{U} \geq 0$.

$$\mathcal{U} = \frac{1}{2}\sigma_{ij}\left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)$$

$$= \frac{1}{2E}\left((1+\nu)\,\sigma_{ij}\sigma_{ij} - \nu\,\sigma_{ii}\,\sigma_{jj}\right).$$

$\mathcal{U} \geq 0$