AS5900

Lecture 29

Strain energy density

$$\mathcal{U}(\underset{\sim}{e}) = \frac{1}{2}\lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} \geq 0$$

$$\underset{\sim}{\mathcal{U}}(\underset{\sim}{\sigma}) = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{E}\sigma_{ii}\sigma_{jj}$$

Next, we will show that

$$\sigma_{ij} = \frac{\partial \mathcal{U}}{\partial e_{ij}}$$

<u>Verify:</u> $\frac{\partial \mathcal{U}}{\partial e_{ij}} = \frac{\partial}{\partial e_{ij}}\left(\frac{\lambda}{2} e_{mm} e_{nn} + \mu e_{mn} e_{mn}\right).$

$\frac{\partial}{\partial e_{12}} \Rightarrow$ vary e_{12}, while keeping all other e_{ij} ($e_{11}, e_{22}, e_{33}, e_{21}, e_{23}$)

$$\frac{\partial U}{\partial e_{ij}} = \frac{\lambda}{2}\left\{\cancel{\frac{\partial e_{mm}}{\partial e_{ij}}}_{\delta_{ij}}\, e_{nn} + e_{mm}\, \cancel{\frac{\partial e_{nn}}{\partial e_{ij}}}_{\delta_{ij}}\right\} +$$

$$\mu\left\{\cancel{\frac{\partial e_{mn}}{\partial e_{ij}}}_{\delta_{mi}\delta_{nj}}\, e_{mn} + e_{mn}\, \cancel{\frac{\partial e_{mn}}{\partial e_{ij}}}_{\delta_{mi}\delta_{nj}}\right\}$$

$$\frac{\partial e_{mm}}{\partial e_{ij}}$$

$m = 3$, $i = 1, j = 2$	$m = 2$, $i = 1, j = 2$	$m=2$, $i=j=2$
$= \frac{\partial e_{33}}{\partial e_{12}}$	$\frac{\partial e_{22}}{\partial e_{12}}$	$\frac{\partial e_{22}}{\partial e_{22}}$
$= 0$	$= 0$	$= 1$

So, $\frac{\partial e_{mm}}{\partial e_{ij}} = \delta_{mi}\,\delta_{mj} = \delta_{ij}$

$$\frac{\partial U}{\partial e_{ij}} = \frac{\lambda}{\cancel{2}}\,\delta_{ij}\,(\cancel{2}\,e_{\cancel{nn}\,kk}) +$$

$$\mu\left\{2 e_{ij}\right\} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$= \sigma_{ij}$ by Hooke's law.

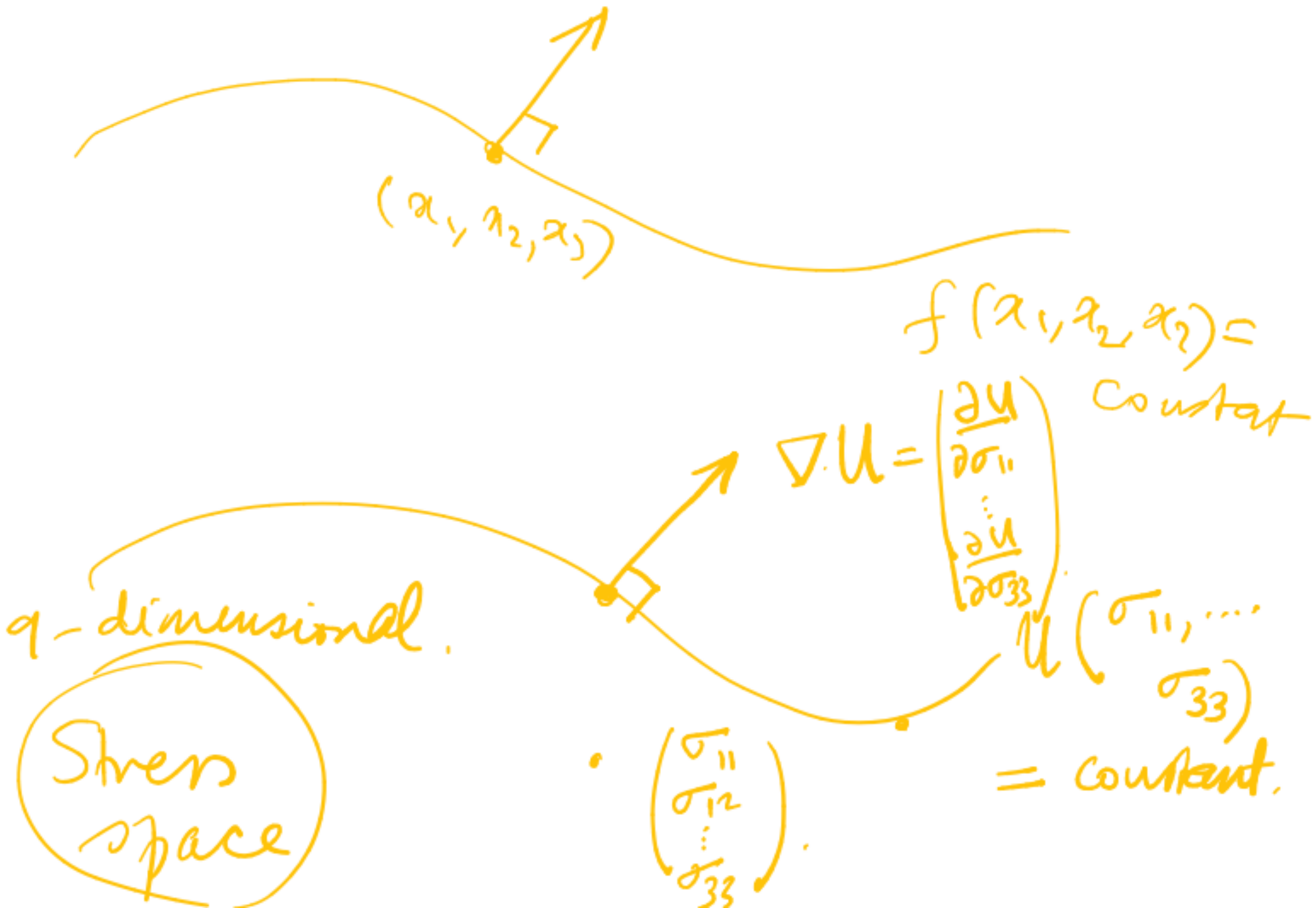
H/W: Show that:

$$e_{ij} = \frac{\partial U}{\partial \sigma_{ij}}$$

"strain is the $\perp$r to the constant U curve in stress space"!

function $f(x_1, x_2, x_3)$

$$\nabla f = \frac{\partial f}{\partial x_i} \underset{\sim}{e_i}$$



" σ_{ij} is the normal to the U = const surface in strain space".

$$\sigma_{ij} = \frac{\partial U}{\partial e_{ij}}$$

$$\frac{\partial \sigma_{ij}}{\partial e_{kl}} = \frac{\partial^2 U}{\partial e_{kl}\,\partial e_{ij}}$$

$$\sigma_{kl} = \frac{\partial U}{\partial e_{kl}}$$

$$\frac{\partial \sigma_{kl}}{\partial e_{ij}} = \frac{\partial^2 U}{\partial e_{ij}\,\partial e_{kl}}$$

compare & find them to be equal.

General anisotropic linear elastic solid ∴

$\sigma_{ij} = C_{ijmn} e_{mn}$ $\quad$ $\sigma_{kl} = C_{klmn} e_{mn}$

$$\frac{\partial \sigma_{ij}}{\partial e_{kl}} = \frac{\partial \sigma_{kl}}{\partial e_{ij}}$$

$$\Rightarrow C_{ijmn} \underbrace{\cancel{\frac{\partial e_{mn}}{\partial e_{kl}}}}_{\delta_{mk}\delta_{nl}} = C_{klmn} \underbrace{\cancel{\frac{\partial e_{mn}}{\partial e_{ij}}}}_{\delta_{mi}\delta_{nj}}$$

$$\Rightarrow \quad C_{ijkl} = C_{klij}$$ "major symmetry of $\underset{\sim}{C}$".

Voight notation:

11	$\rightarrow$	1
22	$\rightarrow$	2
33	$\rightarrow$	3.
12	$\rightarrow$	4
23	$\rightarrow$	5
13	$\rightarrow$	6.

$$C_{1112} = \mathcal{C}_{14}.$$

$$C_{3113} = \mathcal{C}_{66}$$

etc.

C_{1113}

$$\begin{Bmatrix}\sigma_1\\ \vdots \\ \sigma_6\end{Bmatrix} = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \cdots & \mathcal{C}_{16} \\ & \ddots & & \\ & & \ddots & \\ \mathcal{C}_{61} & & & \mathcal{C}_{66}\end{bmatrix}\begin{Bmatrix} e_1 \\ e_2\\ \vdots \\ e_6\end{Bmatrix}$$

C_{1311}

Major symmetry $\Rightarrow \mathcal{C}$ matrix is symmetric.

$\Rightarrow$ Only independent entries of C_6 fall on the diagonal or above.

# of such entries =

$$6+5+4+3+2+1 = 21.$$

$\Rightarrow$ Even in the most anisotropic linear elastic solid there can be only 21 independent C_{ijkl}.

Bounds on elastic constants.

Uniaxial tension: ($E > 0$).

$$[\sigma_{ij}] = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$U = \frac{1+\nu}{2E}\sigma^2 - \frac{\nu}{E}\sigma^2$$

$$= \frac{\sigma^2}{2E} \geq 0$$

we know from end of last lecture.

$\Rightarrow E \geq 0$

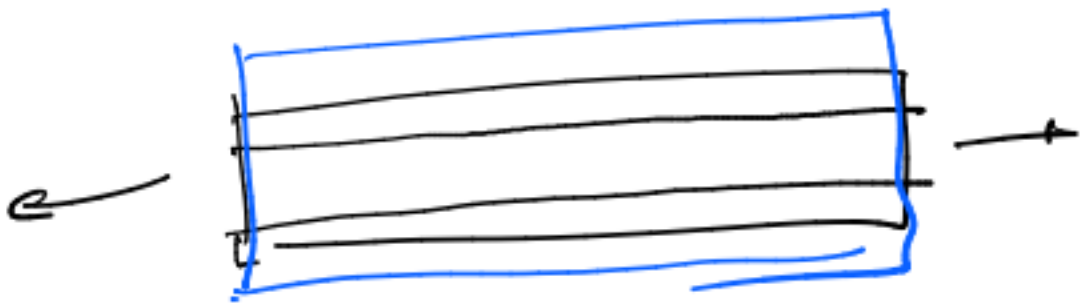
Simple shear ($\nu \geq -1$)

$$\sigma_{ij} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$U = \frac{1+\nu}{2E}\left(2\tau^2\right) - \frac{\nu}{E}\cdot 0 \geq 0$$

$$\Rightarrow \quad 1+\nu \geq 0$$

$$\Rightarrow \quad \boxed{\nu \geq -1.}$$



Hydrostatic pressure $\left(\nu \leq \frac{1}{2}\right)$.

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$U = \frac{1+\nu}{2E}\, 3p^2 - \frac{3\nu}{2E}\left(\overset{3}{\cancel{9}}\, p^2\right) \geq 0.$$

$$= \frac{3p^2}{2E}\left\{1+\nu-3\nu\right\} = \frac{3p^2}{2E}\left(1-2\nu\right) \geq 0$$

$$\Rightarrow \quad 1 - 2\nu \geq 0$$

$$\Rightarrow \boxed{\nu \leq \frac{1}{2}}$$

$$\Rightarrow \underbrace{\boxed{E \geq 0} \quad \& \quad \boxed{-1 \leq \nu \leq \frac{1}{2}}}_{\text{follow from } \mu \geq 0.}$$