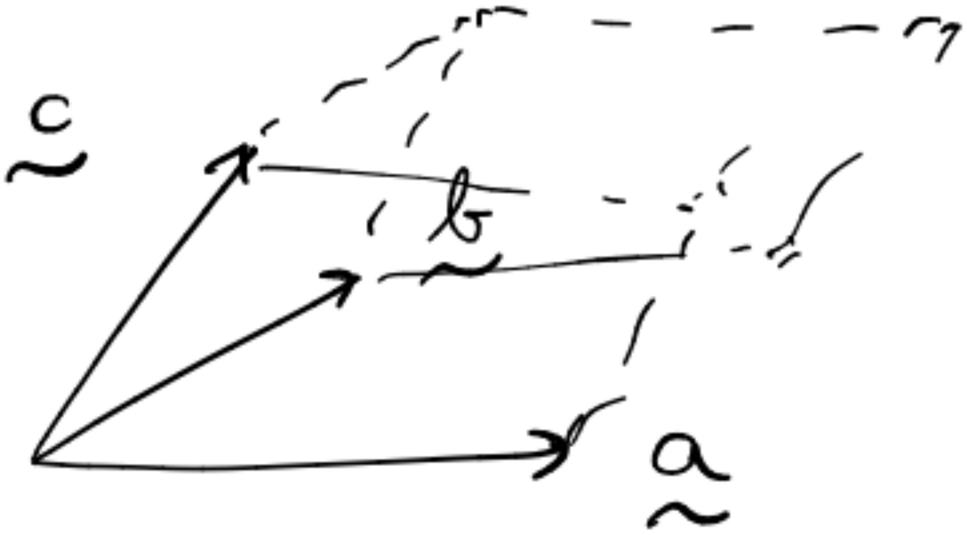
Lecture 3

"bac – cab rule":

$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b}(\underset{\sim}{a} \cdot \underset{\sim}{c}) - \underset{\sim}{c}(\underset{\sim}{a} \cdot \underset{\sim}{b}).$$



prove by usual method:

$$\underset{\sim}{a} = a_1 \underset{\sim}{e_1} + a_2 \underset{\sim}{e_2} + a_3 \underset{\sim}{e_3} \qquad \text{etc}$$

$$\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \ldots \text{ very tedious.}$$

prove using Indicial notation.

$$(\underset{\sim}{b} \times \underset{\sim}{c})_{\cancel{i}\,k} = \varepsilon_{\cancel{i}\cancel{j}\cancel{k}\;k\,p\,q}\, b_{\cancel{j}\,p}\, c_{\cancel{k}\,q}.$$

$$(\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}))_i = \varepsilon_{ijk}\, a_j \underbrace{\varepsilon_{kpq}\, b_p\, c_q}_{k\text{-th comp of } \underset{\sim}{b} \times \underset{\sim}{c}}.$$

$$(\underset{\sim}{a}\times(\underset{\sim}{b}\times\underset{\sim}{c}))_i = \varepsilon_{ijk}\,\varepsilon_{kpq}\, a_j\, b_p\, c_q .$$

$$\varepsilon_{ijk}\ \varepsilon_{\substack{\cancel{pqr}\\ rpq}} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}.$$

$$\varepsilon_{pqr} = \varepsilon_{qrp} = \varepsilon_{rpq}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312}$$

put $r \leftarrow k$

$$\varepsilon_{ijk}\,\varepsilon_{kpq} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{vmatrix}$$

$$\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3.$$

$$\varepsilon_{ijk}\,\varepsilon_{kpq} = \begin{vmatrix} \delta_{ip}^{+} & \delta_{iq} & \delta_{ik} \\ \delta_{jp}^{-} & \delta_{jq} & \delta_{jk} \\ \delta_{kp}^{+} & \delta_{kq}^{-} & 3^{+} \end{vmatrix}.$$

$$= \underline{\delta_{kp}}\left(\delta_{iq}\,\underline{\delta_{jk}} - \delta_{jq}\,\delta_{ik}\right)$$
$$- \delta_{kq}\left(\delta_{ip}\,\delta_{jk} - \delta_{ik}\,\delta_{jp}\right)$$
$$+ 3\left(\delta_{ip}\,\delta_{jq} - \delta_{jp}\,\delta_{iq}\right).$$

contraction

$$= \delta_{jp}\,\delta_{iq} - \delta_{ip}\,\delta_{jq}$$
$$- \delta_{jq}\,\delta_{ip} + \delta_{iq}\,\delta_{jp}$$
$$+ 3\left(\delta_{ip}\,\delta_{jq}\right) - 3\left(\delta_{jp}\,\delta_{iq}\right).$$

$$= -\delta_{jp}\,\delta_{iq} + \delta_{jq}\,\delta_{ip}.$$

$$\left(\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})\right)_i = \varepsilon_{ijk}\,\varepsilon_{kpq}\, a_j\, b_p\, c_q$$

$$= \left(\delta_{jq}\delta_{ip} - \delta_{iq}\delta_{jp}\right) a_j b_p c_q$$

$$= \delta_{jq}\delta_{ip}\, a_j\, b_p\, c_q$$

$$- \delta_{iq}\delta_{jp}\, a_j\, b_p\, c_q$$

Contraction

$$= a_q\, b_i\, c_q - c_i\, a_p\, b_p.$$

$$= b_i\, \underbrace{(a_q c_q)}_{\underset{\sim}{a}\cdot\underset{\sim}{c}} - c_i\, \underbrace{(a_p b_p)}_{\underset{\sim}{a}\cdot\underset{\sim}{b}}.$$

$$\Rightarrow \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b}\,(\underset{\sim}{a}\cdot\underset{\sim}{c}) - \underset{\sim}{c}\,(\underset{\sim}{a}\cdot\underset{\sim}{b}).$$

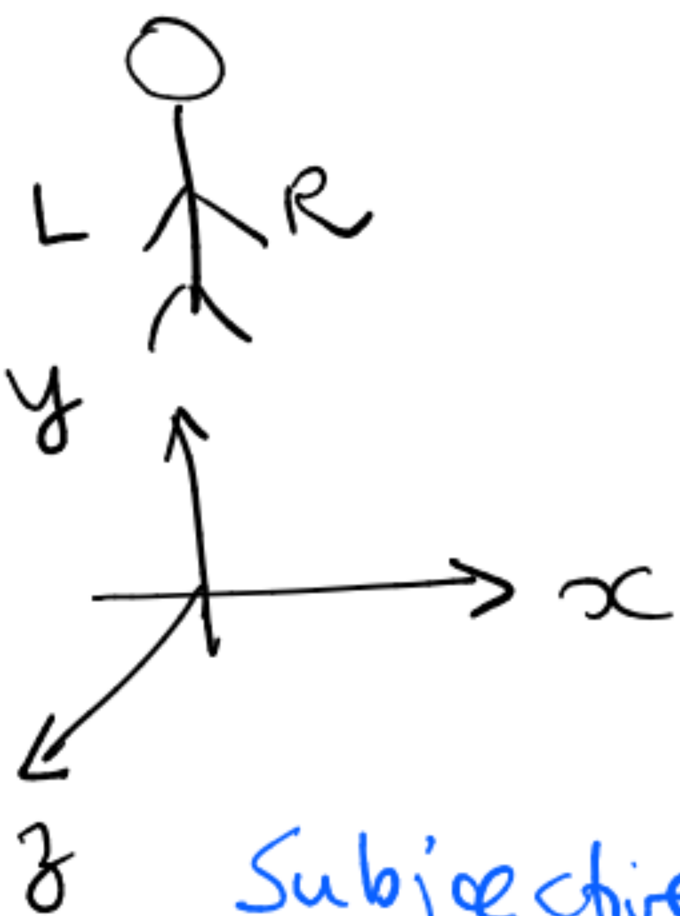
§ Sec 1.4 Coordinate transformations

Physical process | Observed process.

Objective

"Water flows along $+g$".

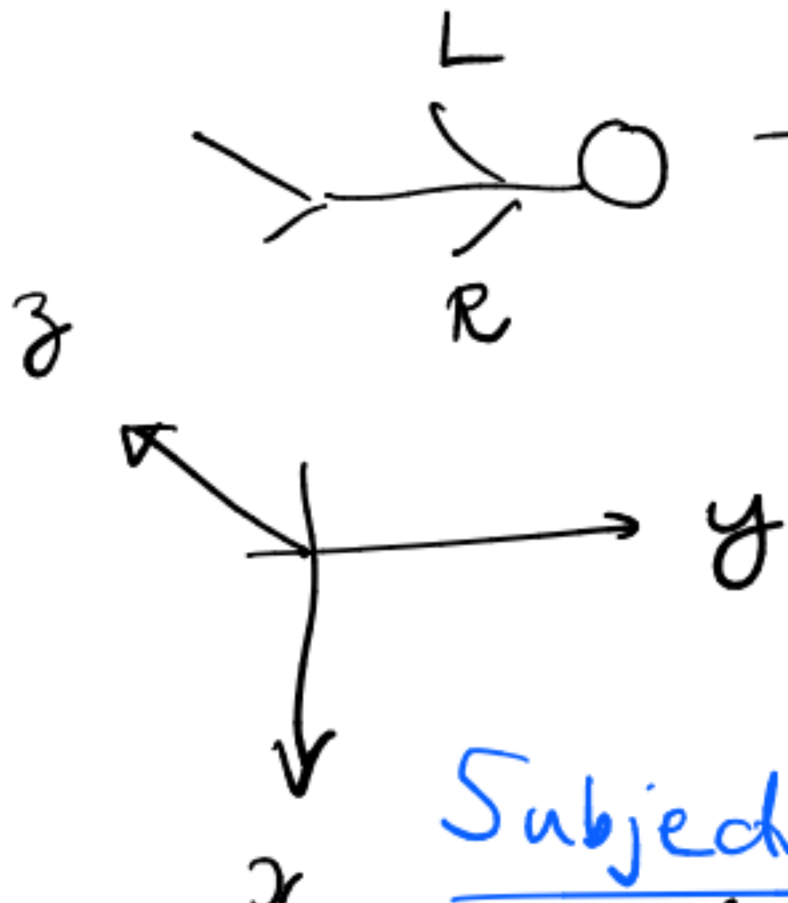
Observer 1



Subjective.

"Water flows along the $-y$ direction".

Observer 2



Subjective

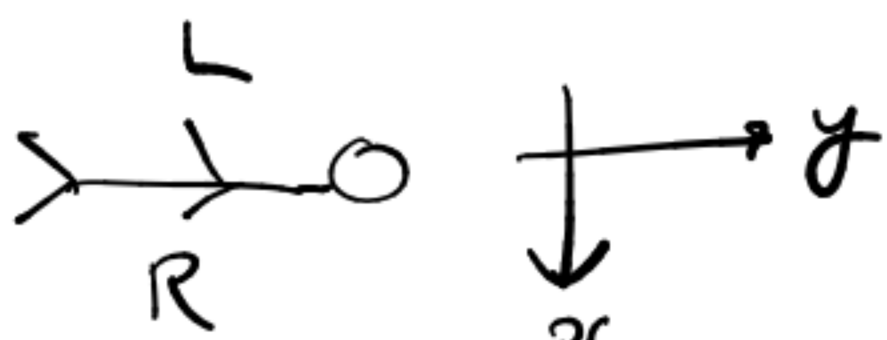
"Water flows along the $+x$ direction".

Observer 1	Observer 2
Velocity vector	Velocity vector
$\underset{\sim}{v} = -v\,\underset{\sim}{j}$	$\underset{\sim}{v} = v\,\underset{\sim}{i}$.

Vectorial representation are subjective.

Objective fact needs a coordinate system to represent.

"$\sigma_{yy} = \frac{T}{A}$"



"$\sigma_{xx} = T/A$".

$$\underbrace{\underset{\uparrow \atop \text{matrix}}{\begin{bmatrix}0&0&0\\0&T/A&0\\0&0&0\end{bmatrix}} \quad \Bigg| \quad \underset{\uparrow \atop \text{matrix.}}{\begin{bmatrix}T/A&0&0\\0&0&0\\0&0&0\end{bmatrix}}}$$

"representation of an objective stress in a certain coordinate system".

tensor.

eg stress

Representation of a tensor in a coordinate system is a scalar/vector/matrix/higher order matrix.

Coordinate transformation

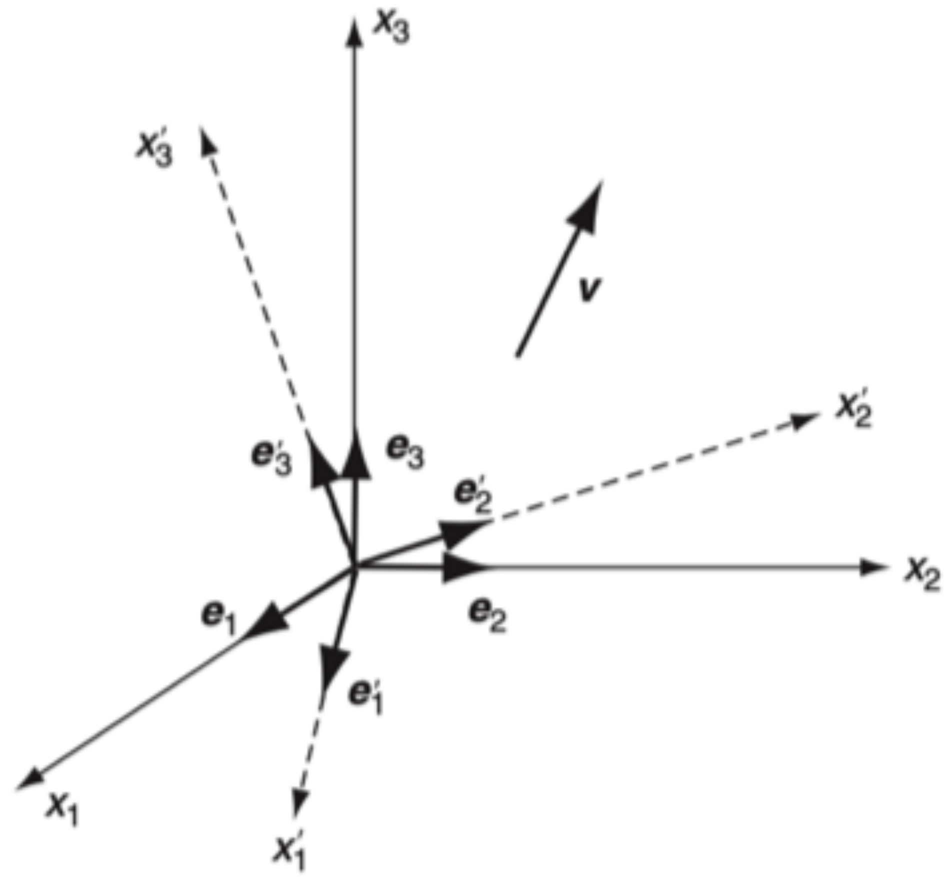


FIGURE 1-1 *Change of Cartesian coordinate frames.*

$x_1 \, x_2 \, x_3$ — one observer

$x_1' \, x_2' \, x_3'$ — another observer

$$Q_{ij} = \cos(x_i', x_j)$$

$$= \underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j .$$

$$\underset{\sim}{e}_i' = Q_{ij} \, \underset{\sim}{e}_j .$$

$$\Rightarrow \underset{\sim}{e}_i = Q_{ji} \, \underset{\sim}{e}_j' .$$