AS5900

Lecture 30

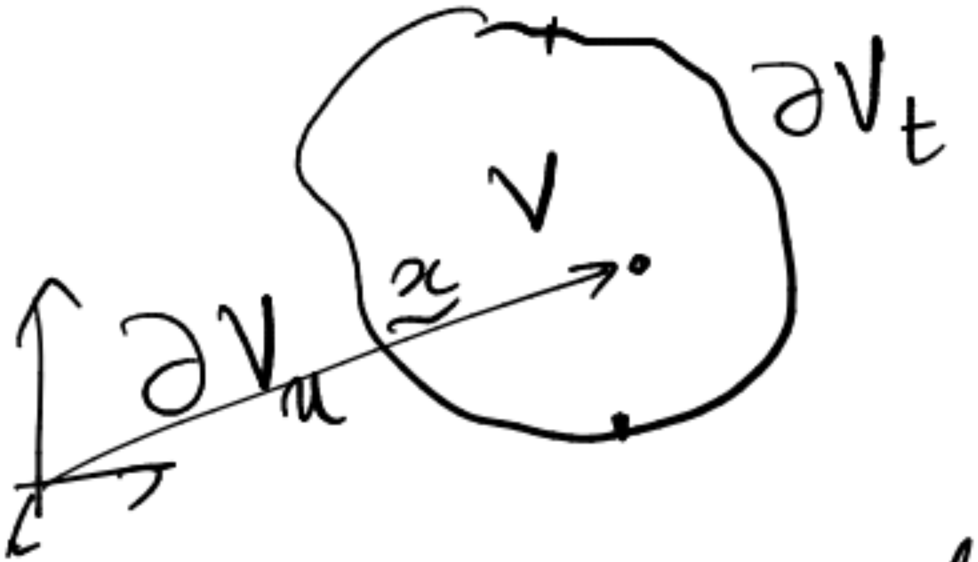
Theorems on linear elasticity:

1. Uniqueness of elasticity boundary value problems.

   "A solution to an EBVP is the solution".

2. Clapeyron's theorem.
3. Reciprocal theorem
4. Principle of virtual work
5. Principles of min. potential energy & complementary potential energy.

1. "A solution to an elastic BVP is the solution".



An Elastostatic state is a set of 3 functions

$$\{ \underset{\sim}{u}(\underset{\sim}{x}), \underset{\sim}{\varepsilon}(\underset{\sim}{x}), \underset{\sim}{\sigma}(\underset{\sim}{x}) \}$$

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

strain-displacement relation.

which obey ① the strain-displacement relation.

② The linear elastic Hooke's law.

③ Governing equations of linear elasticity.
(Navier / Beltrami-Mitchell)

④ Boundary conditions on ∂V_t & ∂V_u.

Such an elastostatic state is called a solution to the elastic Boundary Value Problem

Obj. is to prove that if $\{ \underset{\sim}{u}^{(1)}(\underset{\sim}{x}), \underset{\sim}{\varepsilon}^{(1)}(\underset{\sim}{x}), \underset{\sim}{\sigma}^{(1)}(\underset{\sim}{x}) \}$ is a solution to an elastic BVP there is no other solution.

<u>Proof by contradiction</u>:

Let there be another elastostatic state $\{ \underset{\sim}{u}^{(2)}(\underset{\sim}{x}), \underset{\sim}{\varepsilon}^{(2)}(\underset{\sim}{x}), \underset{\sim}{\sigma}^{(2)}(\underset{\sim}{x}) \}$.

We must show that:

$$\Delta \underset{\sim}{u} = \underset{\sim}{u}^{(2)} - \underset{\sim}{u}^{(1)} \equiv 0.$$

$$\Delta \underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^{(2)} - \underset{\sim}{\varepsilon}^{(1)} \equiv 0.$$

$$\Delta \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{(2)} - \underset{\sim}{\sigma}^{(1)} \equiv 0,$$

$\underset{\sim}{\sigma}^{(2)}$ & $\underset{\sim}{\sigma}^{(1)}$ are both solution to EBVP. So

$$\sigma^{(1)}_{ij,j} + b_i = 0$$

$$\sigma^{(2)}_{ij,j} + b_i = 0.$$

$$\Delta \sigma_{ij,j} = 0 \quad \text{(A)}$$

(bi does not appear).

On ∂V_t, prescribed.

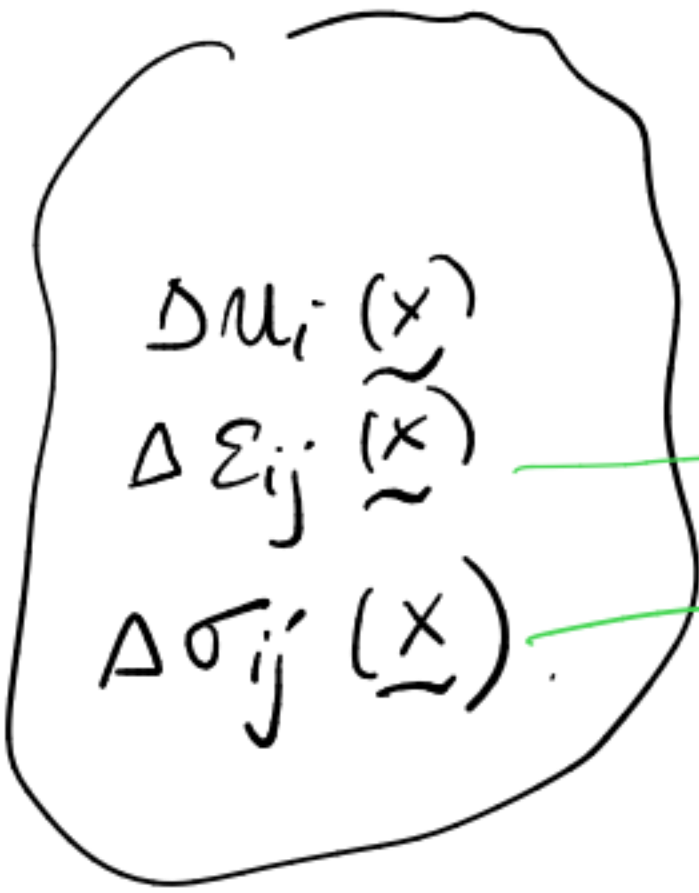
$$\sigma^{(1)}_{ij} n_j = \underset{\sim}{t}_0^{(n)}$$

$$\sigma^{(2)}_{ij} n_j = \underset{\sim}{t}_0^{(n)}$$

Subtracting.

$$\Delta\sigma_{ij}\, n_j = 0 \quad \text{on } \partial V_t \qquad B$$

$||^{ly}$, $$\Delta u_i = 0 \quad \text{on } \partial V_u. \qquad C$$



Calculate strain energy of this solid.

$$\int_V U\, dV = \frac{1}{2}\int_V (\Delta\sigma_{ij})(\Delta\varepsilon_{ij})\, dV$$

$$= \frac{1}{2}\int_V \overbrace{(\Delta\sigma_{ij})}^{\text{symmetric}}\left(\Delta u_{i,j} - \Delta\omega_{ij}\right) dV$$

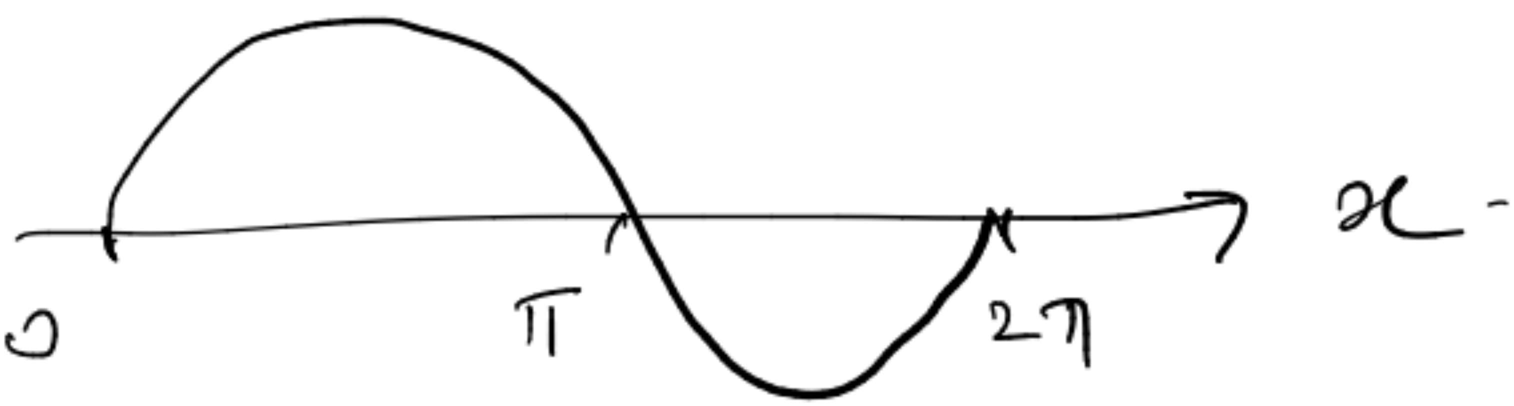
$$= \frac{1}{2}\int_V \Delta\sigma_{ij}\, \Delta u_{i,j}\, dV$$

$$= \frac{1}{2}\left\{ \int_V \overbrace{(\Delta\sigma_{ij}\,\Delta u_i)}^{\Delta P_j}{}_{,j}\, dV - \int_V \Delta\sigma_{ij,j}\,\Delta u_i\, dV \right\}$$

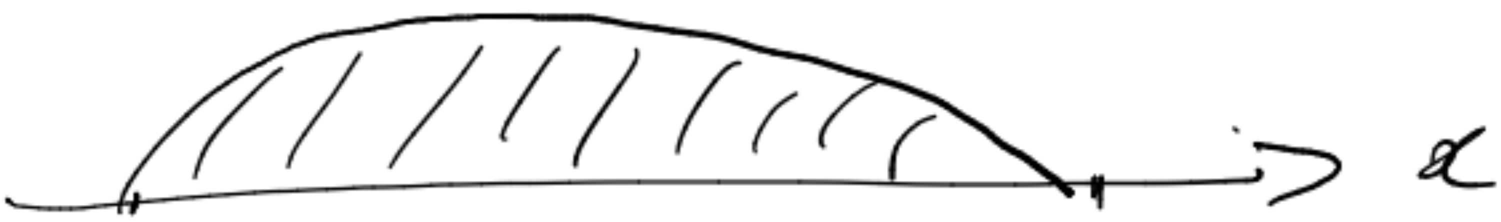
$$= \frac{1}{2}\left\{ \int_{\partial V} \Delta\sigma_{ij}\, n_j \Delta u_i\, dS - \int_V \Delta\sigma_{ij,j}\,\Delta u_i\, dV \right\}$$

$$= \frac{1}{2}\left\{ \int_{\partial V_t} \overbrace{\Delta\sigma_{ij}\, n_j}^{=0 \text{ by } B} \Delta u_i\, dS + \int_{\partial V_u} \Delta\sigma_{ij} n_j \overbrace{\Delta u_i}^{=0 \text{ by } C}\, dS - \int_V \underbrace{\Delta\sigma_{ij,j}}_{=0\ A}\,\Delta u_i\, dV \right\}.$$

$$\Rightarrow \boxed{\int_V u\, dV = 0.}$$



We know $U \geq 0$.



The only way $\int_V U\,dV = 0$ is if $U = 0$ everywhere in V.

$$U = 0$$

$$\Rightarrow \frac{1}{2}\lambda\left(\Delta e_{xx} + \Delta e_{yy} + \Delta e_{zz}\right)^2 + \mu\left(\Delta e_{xx}^2 + \Delta e_{yy}^2 + \Delta e_{zz}^2 + \frac{1}{2}\Delta\gamma_{xy}^2 + \frac{1}{2}\Delta\gamma_{yz}^2 + \frac{1}{2}\Delta\gamma_{zx}^2\right) = 0.$$

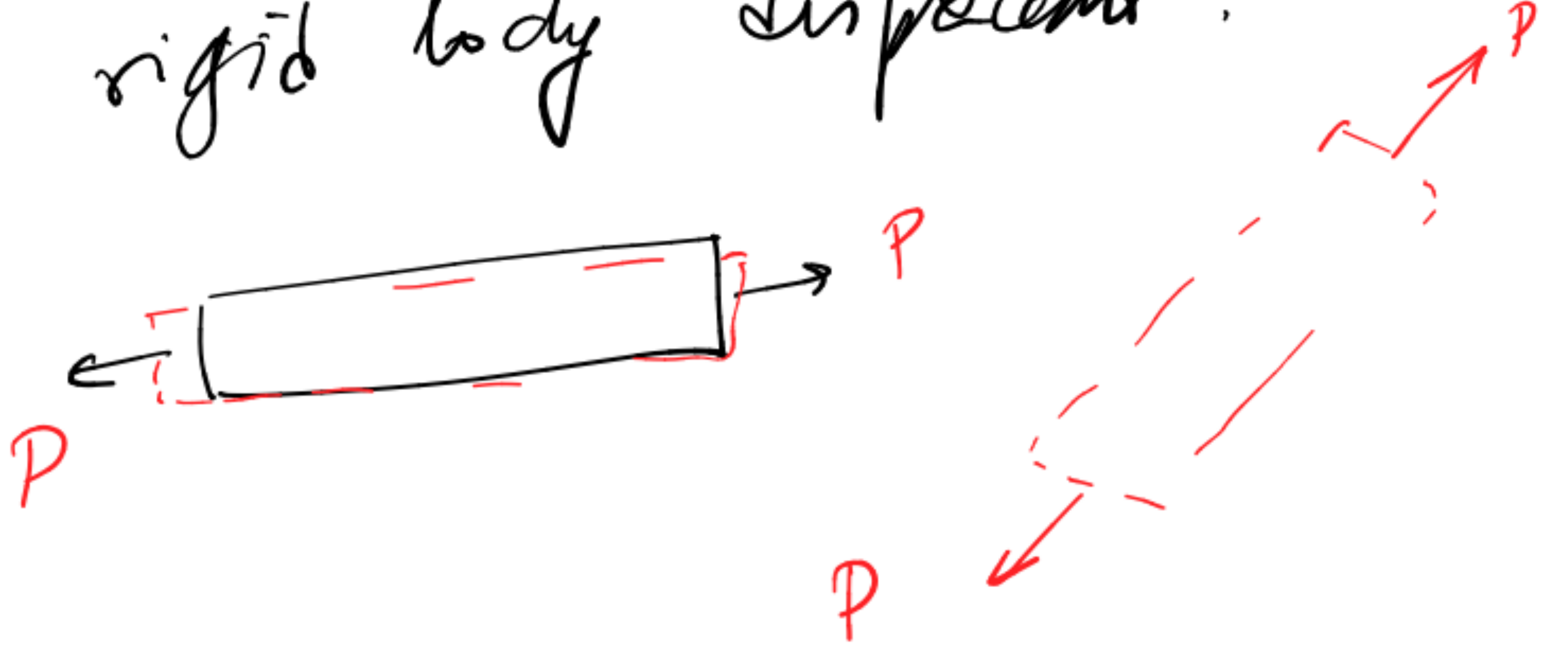
If any $\Delta \varepsilon_{ij}(\underset{\sim}{x})$ were non-zero, the above equality will break down.

This means that

$$\Delta \varepsilon_{ij} \equiv 0 \quad \text{for all } i,j$$

$$\Rightarrow \Delta \sigma_{ij} \equiv 0 \quad \text{" \;\; " \;\; " \;\; "}$$

$\Delta \underset{\sim}{u}$ can however be nonzero, provided it corresponds to a rigid body displacement.

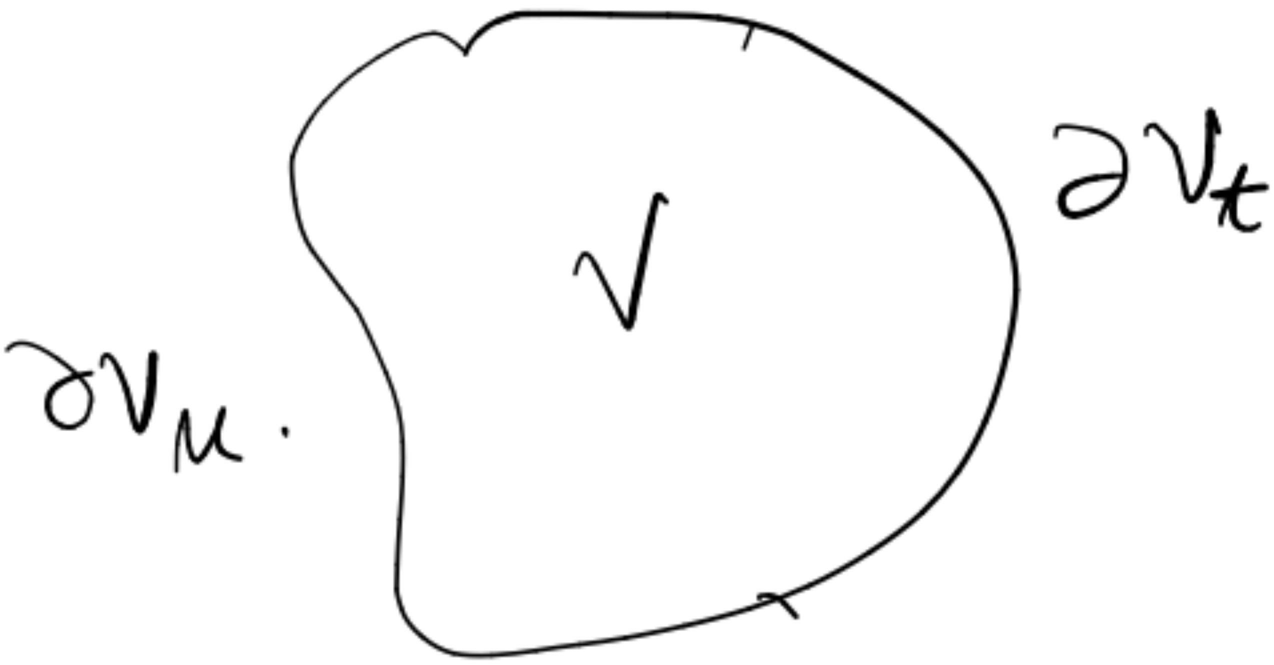


$$\underset{\sim}{\sigma}^{(1)}(\underset{\sim}{x}) \equiv \underset{\sim}{\sigma}^{(2)}(\underset{\sim}{x}) \quad \&$$

$$\underset{\sim}{\varepsilon}^{(1)}(x) \equiv \underset{\sim}{\varepsilon}^{(2)}(\underset{\sim}{x}).$$

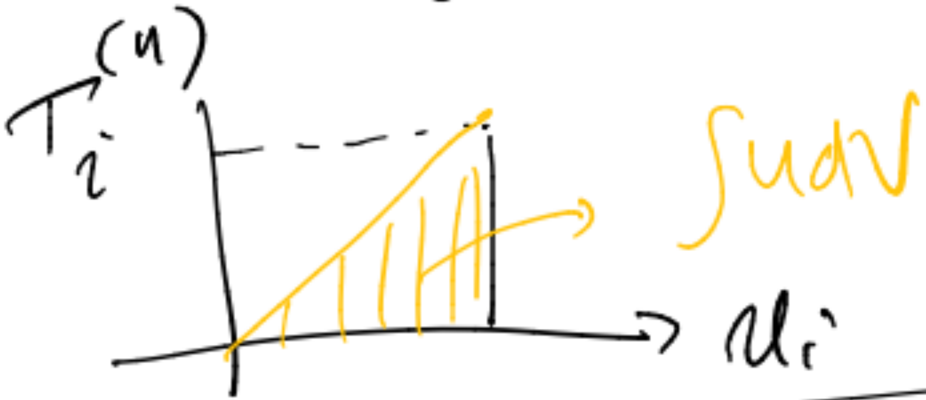
Clapeyron's theorem:

Consider an elastic solid in equilibrium loaded by surface tractions & body forces.



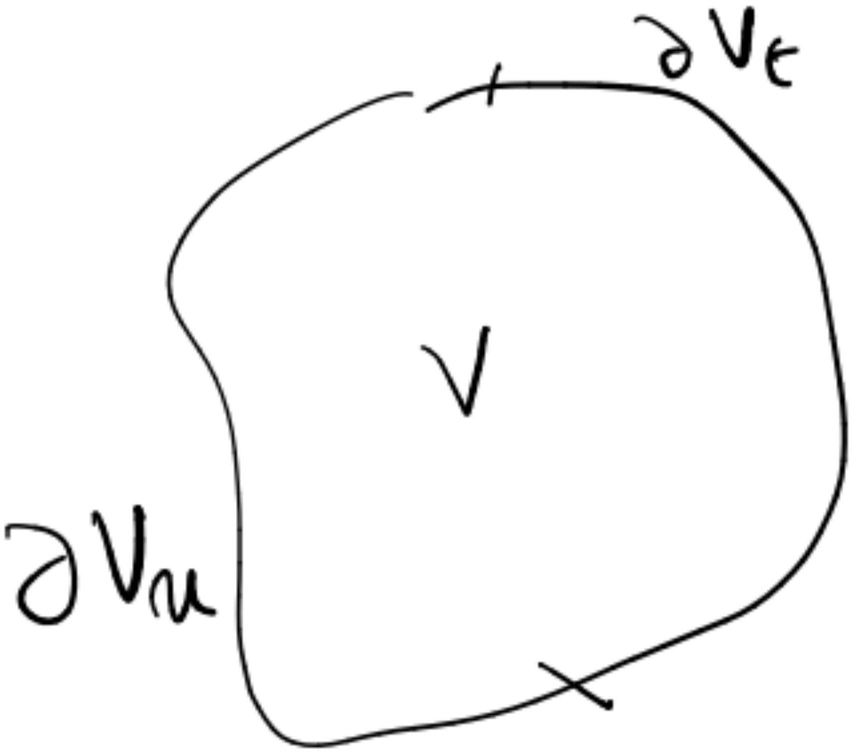
The work by the surface tractions & body forces is twice the strain energy in the body.

$$\Rightarrow \int_{\partial V_t} T_i^{(n)} u_i \, dS + \int_V b_i u_i \, dV = 2 \int_V U \, dV .$$



Note: factor of 2 on the RHS arises because of quasistatic loading.

Reciprocal theorem:



2 sets of surface traction & body forces.

$\{ \underset{\sim}{T}^{(1)}, \underset{\sim}{b}^{(1)} \}$

$\{ \underset{\sim}{T}^{(2)}, \underset{\sim}{b}^{(2)} \}$.

Let $\underset{\sim}{u}^{(i)}(\underset{\sim}{x})$ be the displacement field corresponding loading $\{\underset{\sim}{T}^{(i)}, \underset{\sim}{b}^{(i)}\}$, $i = 1, 2$.

Reciprocal theorem:

Work done by $\{\underset{\sim}{T}^{(1)}, \underset{\sim}{b}^{(1)}\}$ on $\underset{\sim}{u}^{(2)}$

$=$

Work done by $\{\underset{\sim}{T}^{(2)}, \underset{\sim}{b}^{(2)}\}$ on $\underset{\sim}{u}^{(1)}$.

Or mathemathically:

$$\int_{\partial V} T_i^{(1)} u_i^{(2)} \, dS + \int_V b_i^{(1)} u_i^{(2)} \, dV =$$

$$\int_{\partial V} T_i^{(2)} u_i^{(1)} \, dS + \int_V b_i^{(2)} u_i^{(1)} \, dV.$$

$\rightarrow$ To prove.