AS5900

Lecture 4

Coordinate transformations.

v is an objective quantity.

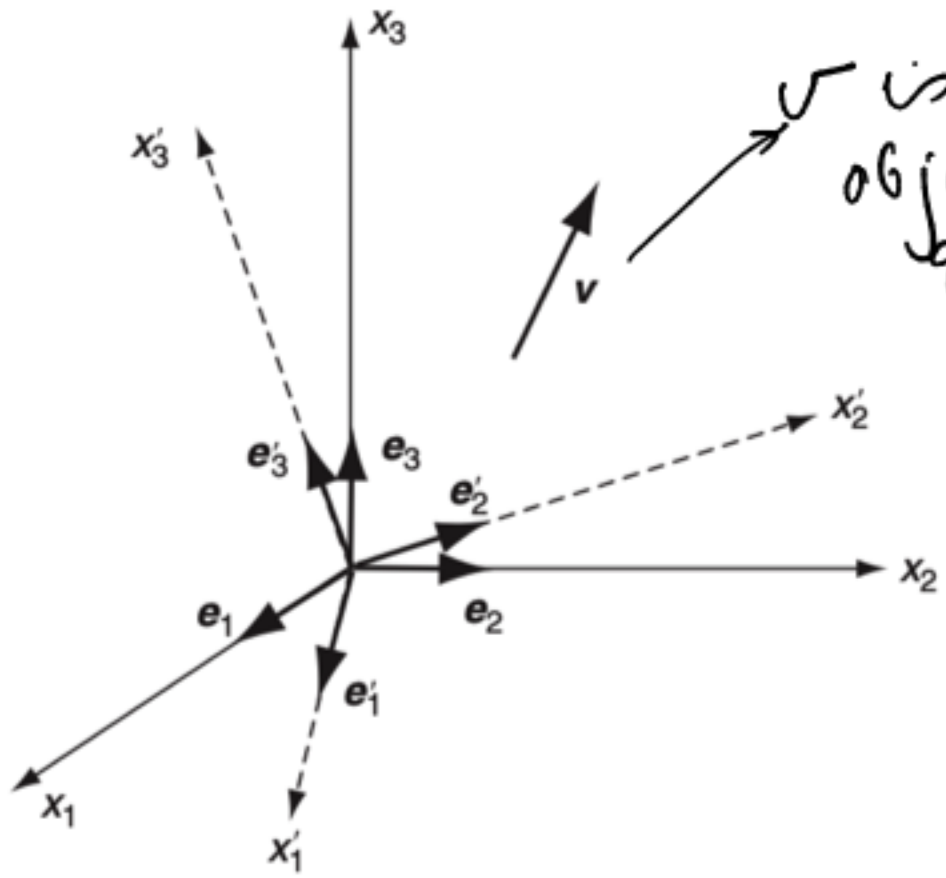


FIGURE 1-1 *Change of Cartesian coordinate frames.*

$\underset{\sim}{v}$ — objective tensor — can be expressed in terms of components (v_1, v_2, v_3) only if we introduce a coordinate system.

(v_1, v_2, v_3) are called the components of $\underset{\sim}{v}$ in a particular coord. system $x_1 x_2 x_3$.

Another coordinate system $x_1' x_2' x_3'$ will give components (v_1', v_2', v_3').

In general $v_1 \neq v_1'$

$$v_2 \neq v_2'$$

$$v_3 \neq v_3'.$$

But both represent the same dyadic tensor $\underset{\sim}{v}$.

$$\underset{\sim}{v} = v_i \, \underset{\sim}{e_i}$$

$$= v_i' \, \underset{\sim}{e_i}'$$

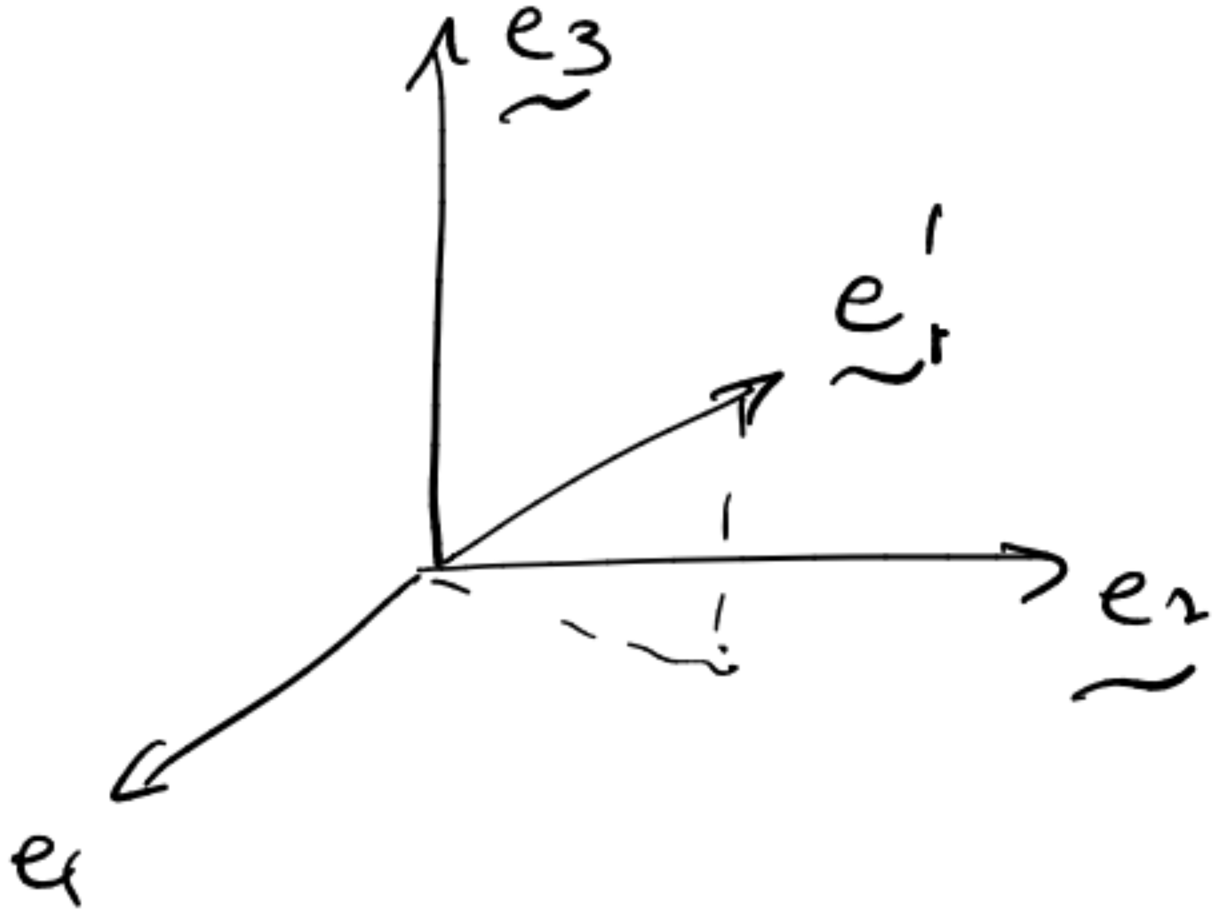
tensor

We want to relate v_i to v_i':

<u>Define:</u> $Q_{ij} = \cos(x_i', x_j)$

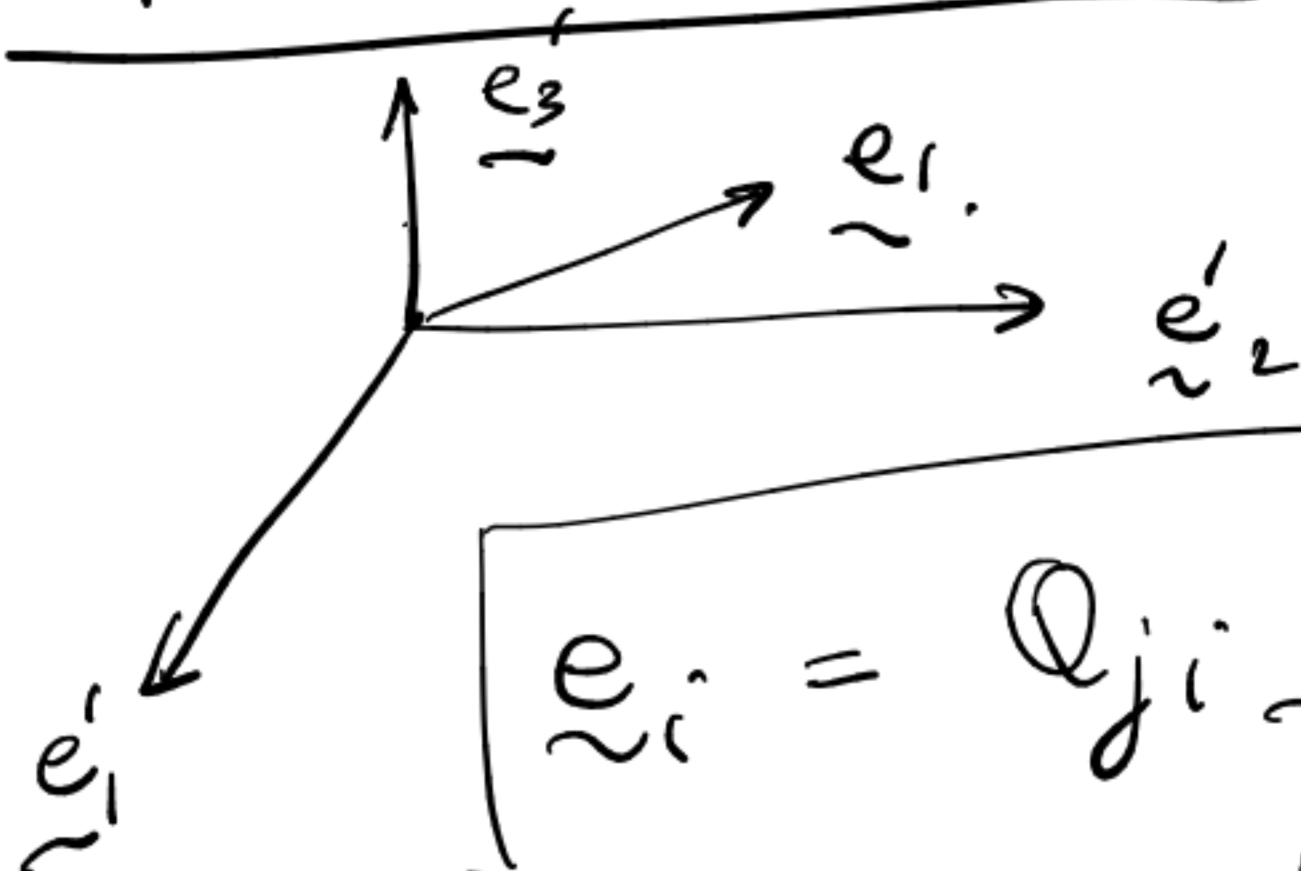
$$= \underset{\sim}{e_i}' \cdot \underset{\sim}{e_j}.$$

$\underset{\sim}{e}_1' =$



$$\underset{\sim}{e}_1' = Q_{11}\,\underset{\sim}{e}_1 + Q_{12}\,\underset{\sim}{e}_2 + Q_{13}\,\underset{\sim}{e}_3.$$

$$\boxed{\underset{\sim}{e}_i' = Q_{ij}\,\underset{\sim}{e}_j.} \quad (1)$$



$$\boxed{\underset{\sim}{e}_i = Q_{ji}\,\underset{\sim}{e}_j'} \quad (2)$$

$$\underset{\sim}{v} = v_i \underset{\sim}{e_i} \overset{②}{=} v_i Q_{ji} \underset{\sim}{e}'_j$$

$$\underset{\sim}{v} = v'_i \underset{\sim}{e}'_i \underset{①}{=} v'_i Q_{ij} \underset{\sim}{e_j}.$$

$$= v'_k Q_{kj} \underset{\sim}{e_j}.$$

$$= v'_k Q_{ki} \underset{\sim}{e_i}.$$

$$\underset{\sim}{v} = v'_j Q_{ji} \underset{\sim}{e_i}.$$

Comparing the 2 circled steps,

$$\boxed{v_i = v'_j Q_{ji}.} \quad ③$$

Compare the 2 boxed steps:

$$v'_i \underset{\sim}{e}'_i = v_i Q_{ji} \underset{\sim}{e}'_j$$

$$= v_j Q_{ij} \underset{\sim}{e}'_i.$$

$$\Rightarrow \boxed{v'_i = v_j Q_{ij}} \quad \dots \quad ④$$

Q_{ij} is a matrix.

It must satisfy $Q_{ji} Q_{jk} = \delta_{ik}$.

proof:

Start from (3):

$$v_i = v'_j Q_{ji}$$

$$\delta_{ik} v_k = \underbrace{v_k Q_{jk}}_{\text{from (4)}} Q_{ji}$$

$$\Rightarrow (Q_{jk} Q_{ji} - \delta_{ik}) v_k = 0$$

for all v_k.

$$\Rightarrow \boxed{Q_{ji} Q_{jk} = \delta_{ik}}$$

To understand this, consider AA^T

$$(AA^T)_{ik} = \sum_{j=1}^{3} A_{ij} A^T_{jk} = A_{ij} A^T_{jk}$$

index $= A_{ij} A_{kj}$.

$$(A^T A)_{ik} = A^T_{ij} A_{jk}$$

$$= A_{ji} A_{jk}.$$

$$(Q^T Q)_{ik} = Q_{ji} Q_{jk} = \delta_{ik}.$$

$$\Rightarrow \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\Rightarrow \boxed{Q^{-1} = Q^T.}$$

Starting from ④ & inserting Eq. ③ into it, we can show that

$$Q Q^T = I$$

$$\Rightarrow \boxed{Q^{-T} = Q}$$

$$\det\left\{ Q Q^T = I \right\}$$

LHS:

$$\det(Q Q^T) = \det(Q)\det(Q^T)$$

$$= \det(Q)\det(Q)$$

$$= \{\det(Q)\}^2$$

$$\|$$

RHS:

$$\det(I) = 1$$

$$\underbrace{\{\det(Q)\}^2}_{\text{LHS}} = \underbrace{1}_{\text{RHS}}$$

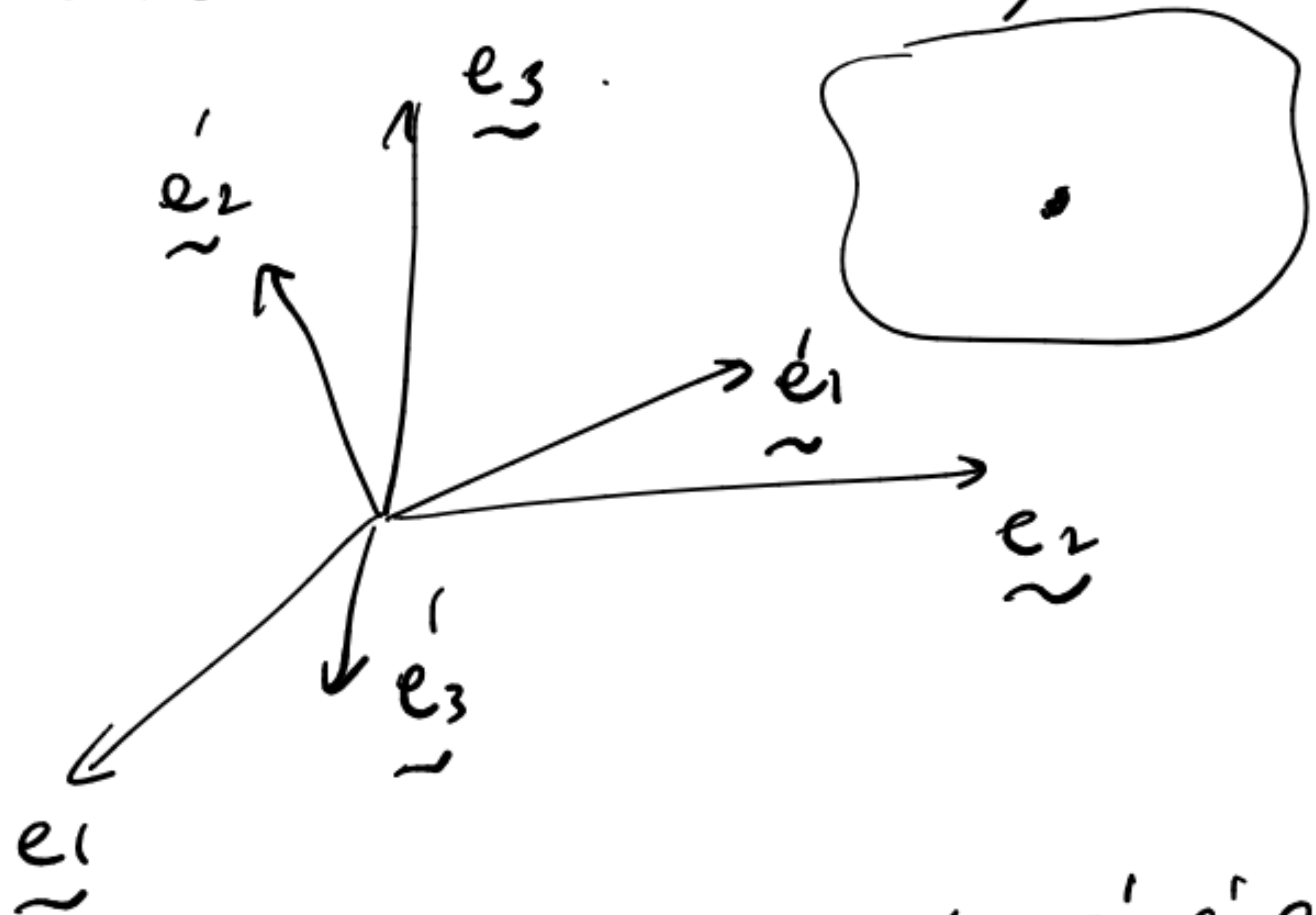
$$\{\det(Q)\}^2 = 1.$$

$$\Rightarrow \boxed{\det(Q) = \pm 1.}$$

Rotations $\longrightarrow$ $\det(Q) = 1$

Reflections $\longrightarrow$ $\det(Q) = -1$.

Suppose the objective quantity is a 2-tensor. (stress, strain)



2 observers $\underset{\sim}{e_1}\,\underset{\sim}{e_2}\,\underset{\sim}{e_3}$ & $\underset{\sim}{e_1'}\,\underset{\sim}{e_2'}\,\underset{\sim}{e_3'}$ measure an objective 2-tensor, $\underset{\sim}{A}$, in their respective coordinate systems.

A_{ij} in xyz — 1^{st} observer.

A'_{ij} in $x'y'z'$ — 2^{nd} observer.

Q_{ij} which relates xyz to $x'y'z'$ is known.

How are A_{ij} & A'_{ij} related?

$$\cancel{\underset{\sim}{v} = v_i \underset{\sim}{e_i}}$$

$$\underset{\sim}{A} = A_{ij} \, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$$

objective 2-tensor

$$\underset{\sim}{e_1} \otimes \underset{\sim}{e_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

etc.

$$= A'_{ij} \, \underset{\sim}{e_i}' \otimes \underset{\sim}{e'_j}$$