# AS5900

## Lecture 5.

Objective ~~vector~~ 1-tensor $\underset{\sim}{v}$ has subjective representation in terms of components in particular coordinate system $xyz$ as a vector. $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

In another coord. system, $x'y'z'$ it will have other components $\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix}$.

$$Q_{ij} = \underset{\sim}{e_i}' \cdot \underset{\sim}{e_j}.$$

$Q_{ij}$ relates $v_i$ to $v_i'$:

$$v_i' = Q_{ij} v_j \text{ and}$$

$$v_i = Q_{ji} v_j'$$

An objective 2-tensor $\underset{\sim}{A}$ has subjective representation in terms of components $a_{ij}$ in coordinate system $xyz$ and $a'_{ij}$ in coordinate system $x'y'z'$.

$$Q_{ij} = \underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j.$$

$$\underset{\sim}{A} = a_{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j. \quad \cdots \quad \nabla$$

objective

$$= a'_{ij}\, \underset{\sim}{e}'_i \otimes \underset{\sim}{e}'_j. \quad \Delta$$

2 - tensor

Want to find how $a_{ij}$ & $a'_{ij}$ are related.

$$\underset{\sim}{e}'_i = Q_{ij}\, \underset{\sim}{e}_j \quad \cdots\cdots \quad (1)$$

$$\underset{\sim}{e}_i = Q_{ji}\, \underset{\sim}{e}'_j \quad \cdots\cdots \quad (2)$$

Substitute (2) into $\nabla$.

$$\underset{\sim}{A} = a_{ij} \left( Q_{ji} \underset{\sim}{e'_j} \right) \otimes \left( Q_{kj} \underset{\sim}{e'_k} \right).$$

$$= a_{ij} Q_{ji} Q_{kj} \quad \underset{\sim}{e'_j} \otimes \underset{\sim}{e'_k}.$$

$$\overset{\Delta}{=} \quad a'_{ij} \quad \underset{\sim}{e'_i} \otimes \underset{\sim}{e'_j}.$$

rename indices

$p \leftarrow i \ , \ i \leftarrow j$

$$= a_{pi} Q_{ip} Q_{ki} \underset{\sim}{e'_i} \otimes e'_k.$$

rename $q \leftarrow j \ , \ j \leftarrow k$

$$= a_{pi} Q_{ip} Q_{ji} \underset{\sim}{e'_i} \otimes \underset{\sim}{e'_j}.$$

messed up beyond repair

... redo

$$\underset{\sim}{A} = a_{ij}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}.$$

$$= a_{ij} \left( Q_{pi}\, \underset{\sim}{e'_p} \right) \otimes \left( Q_{qj}\, \underset{\sim}{e'_q} \right)$$

$$= a_{ij}\, Q_{pi}\, Q_{qj}\, \underset{\sim}{e'_p} \otimes \underset{\sim}{e'_q}.$$

comp

$$= a'_{pq}\, \underset{\sim}{e'_p} \otimes \underset{\sim}{e'_q}$$

$$a'_{pq} = a_{ij}\, Q_{pi}\, Q_{qj}$$

$p \leftrightarrow i$

$q \leftrightarrow j$

$$a'_{ij} = a_{pq}\, Q_{ip}\, Q_{jq}$$

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Let $\underset{\sim}{A}$ be an objective 3-tensor

$$\underset{\sim}{A} = a_{ijk}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} \otimes \underset{\sim}{e_k}$$

$\longrightarrow$ components of $\underset{\sim}{A}$ in $xyz$ coordinate system

$$= a'_{ijk}\, \underset{\sim}{e'_i} \otimes \underset{\sim}{e'_j} \otimes \underset{\sim}{e'_k}.$$

$x'y'z'$

$\longrightarrow$ " " "

We want to relate

$a_{ijk}$ & $a'_{ijk}$

H/w: find the relation & show

$$a'_{ijk} = a_{pqr} \, Q_{ip} \, Q_{jq} \, Q_{kr}.$$

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Definition: A $k$-tensor

$A_{i_1 i_2 i_3 \ldots i_k}$ transforms as

$$A'_{i_1 i_2 i_3 \ldots i_k} = A_{p_1 p_2 \ldots p_k} \, Q_{i_1 p_1} \, Q_{i_2 p_2} \cdots \cdots \, Q_{i_k p_k}.$$

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Coordinate transformation formula for arbitrary tensors.

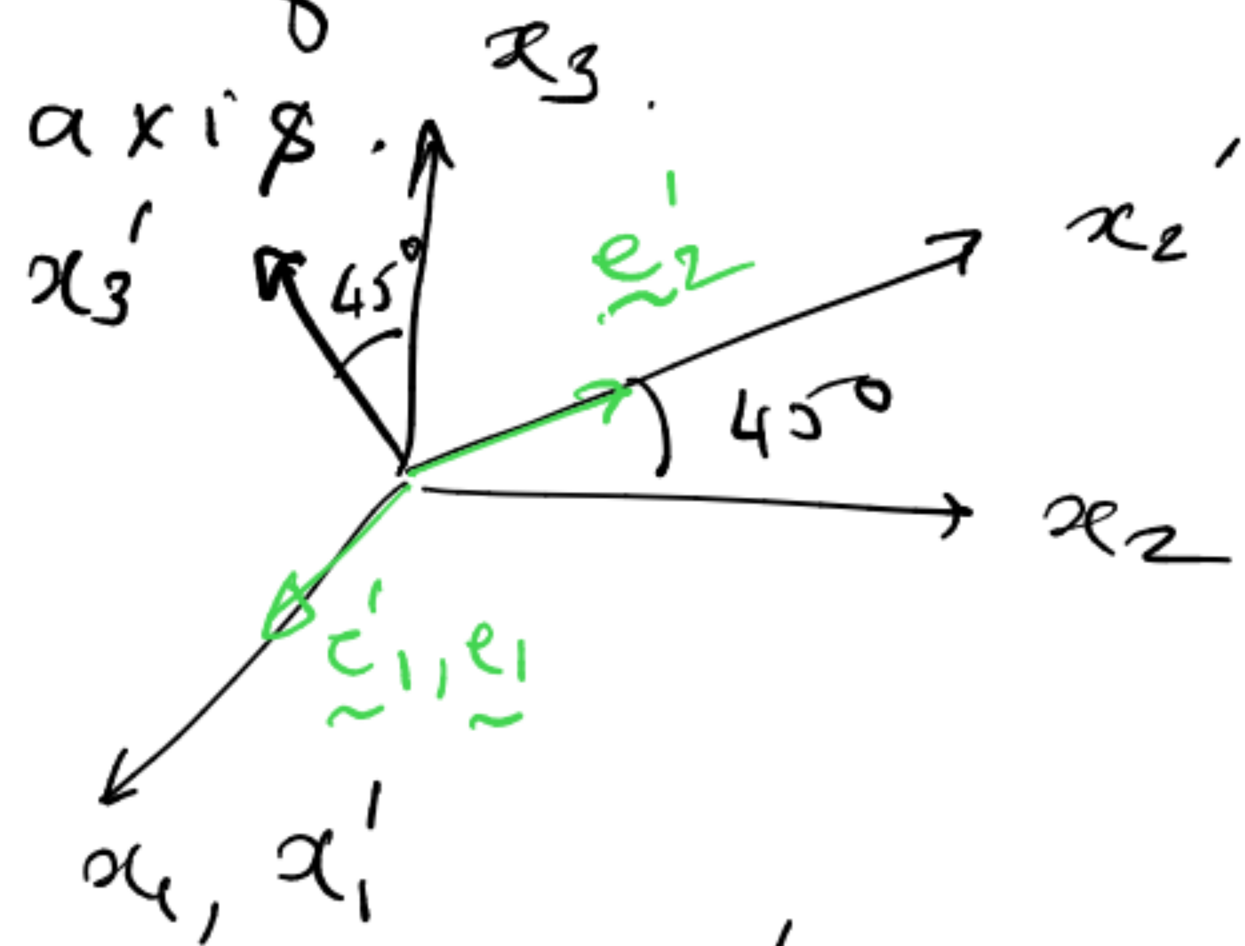
0 tensor is a scalar.

$$A' = A.$$

1-6 Determine components of

$$a_{ij} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix} \text{ in a}$$

new coordinate system through a rotation of $45^\circ$ about the $x_1$-axis.




$$Q_{ij} = ? = \underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j.$$

$$Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$a'_{ij} = a_{pq} \, Q_{ip} \, Q_{jq}.$$

Matrix multiplication:

$$A_{ij} \, B_{jk} = C_{ik}.$$

$$\begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix} \begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix} = \begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}.$$

$$= Q_{ip} \, a_{pq} \, Q^T_{qj}$$

$$a'_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \text{evaluate.}$$