AS5900

Lecture 7

Invariant associated of a 1- tensor:

$$\sqrt{a_i a_i} = \sqrt{a_i' a_i'}.$$

Invariant associated of a 2- tensor.

a_{ij} $\qquad\qquad\qquad$ a_{ij}'

$a_{ij} n_j = \lambda n_i$ $\qquad\qquad$ $a_{ij}' n_j' = \lambda n_i'$

principal directions

$n_i \xleftrightarrow[\text{transform formulas, i.e.}]{\text{coord.}} n_i'$

$$n_i' = Q_{ip} n_p$$

Also, both will obtain λ.

Q: How do the 2 observers calculate their n_i & n_i' & λ?

Obs 1:

$$a_{ij} n_j = \lambda n_i$$
$$= \lambda \delta_{ij} n_j$$

$$\Rightarrow a_{ij} n_j - \lambda \delta_{ij} n_j = 0$$

$$\Rightarrow (a_{ij} - \lambda \delta_{ij}) n_j = 0.$$

$\Rightarrow n_j = 0$ or

$a_{ij} - \lambda \delta_{ij}$ is singular.

$$\Rightarrow \det(a_{ij} - \lambda \delta_{ij}) = 0.$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

Obs 2:

$$a_{ij}' n_j' = \lambda n_i'$$

⋮

$$(a_{ij}' - \lambda \delta_{ij}) n_j' = 0.$$

$n_j' = 0$

$a_{ij}' - \lambda \delta_{ij}$ is singular

$$\det(a_{ij}' - \lambda \delta_{ij}) = 0.$$

copy; put

$a \rightarrow a'$

$$(a_{11}-\lambda)\left((a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}\right)$$

$$-a_{12}\left(a_{21}(a_{33}-\lambda) - a_{31}a_{23}\right)$$

$$+a_{13}\left(a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right) = 0$$

Simplify:

$$\Rightarrow -\lambda^3 + \underline{I}_a\lambda^2 - \underline{II}_a\lambda + \underline{III}_a = 0 \quad (1)$$

$$\underline{I}_a = a_{ii}$$

$$\underline{II}_a = \frac{1}{2}\left(a_{ii}a_{jj} - a_{ij}a_{ij}\right)$$

$$\underline{III}_a = \det\left(a_{ij}\right)$$

put $a \longrightarrow a'$

$$\Rightarrow -\lambda^3 + \underline{I}_{a'}\lambda^2 - \underline{II}_{a'}\lambda + \underline{III}_{a'} = 0. \quad (2)$$

$$\underline{I}_{a'} = a'_{ii}$$

etc

Both xyz & $x'y'z'$ observers must calculate the same λ.

$\Rightarrow$ Eq 1 & Eq 2 must be the same

$\Rightarrow$ The coeffts of Eqs (1) & (2) must match.

$$\Rightarrow \quad \underline{I}_a = a_{ii} = a'_{ii} = \underline{I}_{a'}$$

also: $$\underline{II}_a = \frac{1}{2}\left(a_{ii}a_{jj} - a_{ij}a_{ij}\right)$$

$$= \frac{1}{2}\left(a'_{ii}a'_{jj} - a'_{ij}a'_{ij}\right)$$

$$= \underline{II}_{a'}$$

Finally, $$\underline{III}_a = \det a_{ij} = \det a'_{ij} = \underline{III}_{a'}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & & \\ & & \end{pmatrix}$$

$$a_{ij}a_{ij} = a_{11}^2 + a_{12}^2 + a_{13}^2 + \cdots + a_{33}^2.$$

Solve Eq 1 or Eq. 2 — cubic Eqn.

$\Rightarrow$ 3 roots $\lambda_1, \lambda_2, \lambda_3$.

We will henceforth restrict $\underline{a}$ to symmetric 2 tensors.

$\Rightarrow$ In any coord. system xyz,

$$a_{ij} = a_{ji}.$$

- This implies $\lambda_1, \lambda_2, \lambda_3$ are all real.
- Three possibilities:

I $\lambda_1, \lambda_2, \lambda_3$ distinct.

For each, solve:

$$(a_{ij} - \lambda_1 \delta_{ij}) n_j^{(1)} = 0$$

$\hookrightarrow$ obtain $n_j^{(1)}$.

$$(a_{ij} - \lambda_2 \delta_{ij}) n_j^{(2)} = 0$$

$\hookrightarrow$ obtain $n_j^{(2)}$

$$(a_{ij} - \lambda_3 \delta_{ij}) n_j^{(3)} = 0.$$

$\hookrightarrow$ obtain $n_j^{(3)}$.

$n_i^{(1)}, n_i^{(2)}, n_i^{(3)}$ are the principal directions

Corresp to $\lambda_1, \lambda_2, \lambda_3$ respectively.

- There is a theorem

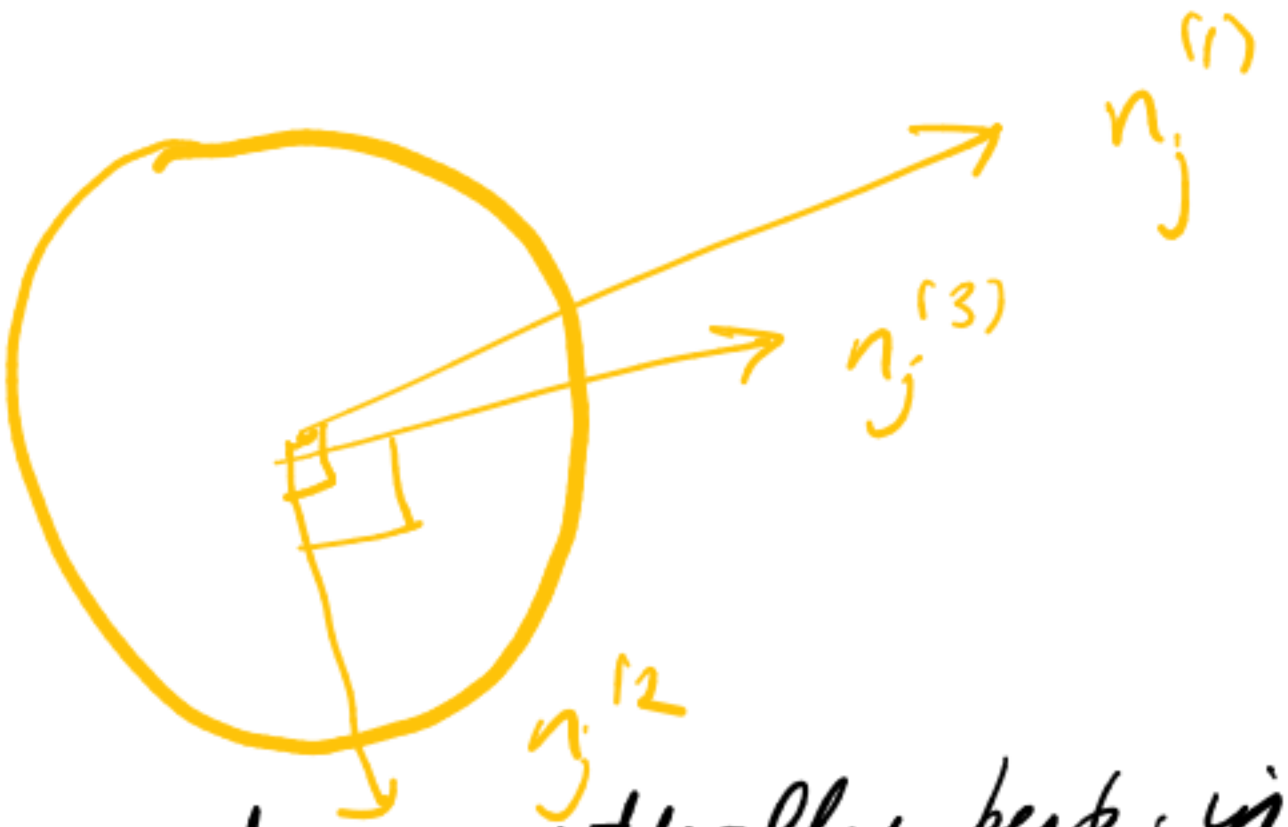
$$n_i^{(1)} n_i^{(2)} = n_i^{(1)} n_i^{(3)} = n_i^{(2)} n_i^{(3)} = 0$$

$\Rightarrow$ The 3 principal dirs are mutually perpendicular.

II $\lambda_1 \neq \lambda_2 = \lambda_3$.

$$(a_{ij} - \lambda_1 \delta_{ij}) n_j^{(1)} = 0.$$

$\hookrightarrow$ Obtain $n_j^{(1)}$.



Any 2 vectors mutually perp. in the plane of normal $n_j^{(1)}$ can be considered the other 2 principal directions.

III $\lambda_1 = \lambda_2 = \lambda_3$.

Eg: $a_{ij} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 3.$$

Any mutually $\perp$ set of 3 vectors can serve as $n_j^{(1)}, n_j^{(2)}, n_j^{(3)}$.

A 2-tensor for which any vector is a principal direction is called an isotropic 2-tensor.

$\lambda_1, \lambda_2, \lambda_3$ are called the principal values of the 2-tensor.

They will remain the same $\forall$ observers.

Eg: $a_{ij} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{pmatrix}$

Find principal values & principal dirns.

$\det(a_{ij} - \lambda \delta_{ij}) = 0$

$-\lambda^3 + 2\lambda^2 + 25\lambda - 50 = 0$

$I_a = 2 + 3 - 3 = 2$

$II_a = \frac{1}{2}(2^2 - 54) = -25$

$III_a = -50.$

$(\lambda-2)(\lambda-5)(\lambda+5) = 0$

$\lambda_1 = 2; \quad \lambda_2 = 5; \quad \lambda_3 = -5.$

$(a_{ij} - \lambda_1 \delta_{ij})\, n^{(1)} = 0.$

$$\begin{pmatrix} \cancel{2-2}\; 0 & 0 & 0 \\ 0 & \cancel{3-2}\; 1 & 4 \\ 0 & 4 & \cancel{-3-2}\; -5 \end{pmatrix} \begin{pmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\underline{2^{nd} \text{ eqn:}}$ $n_2^{(1)} + 4 n_3^{(1)} = 0$

$\rightarrow \; n_2^{(1)} = -4 n_3^{(1)}$.

$\underline{3^{rd} \text{ eqn:}}$ $\cancel{4} n_2^{(1)} = \frac{5}{4} n_3^{(1)}$.

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} n_2^{(1)} = n_3^{(1)} = 0.$

$n_1^{(1)} = \underbrace{\text{arbitrary.}}_{}$

arbitrary constant

$$\begin{pmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\parallel^{rly}$, find $n_j^{(2)} = \alpha_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$$n_j^{(3)} = \alpha_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

verify they are mutually $\underline{\perp}$.