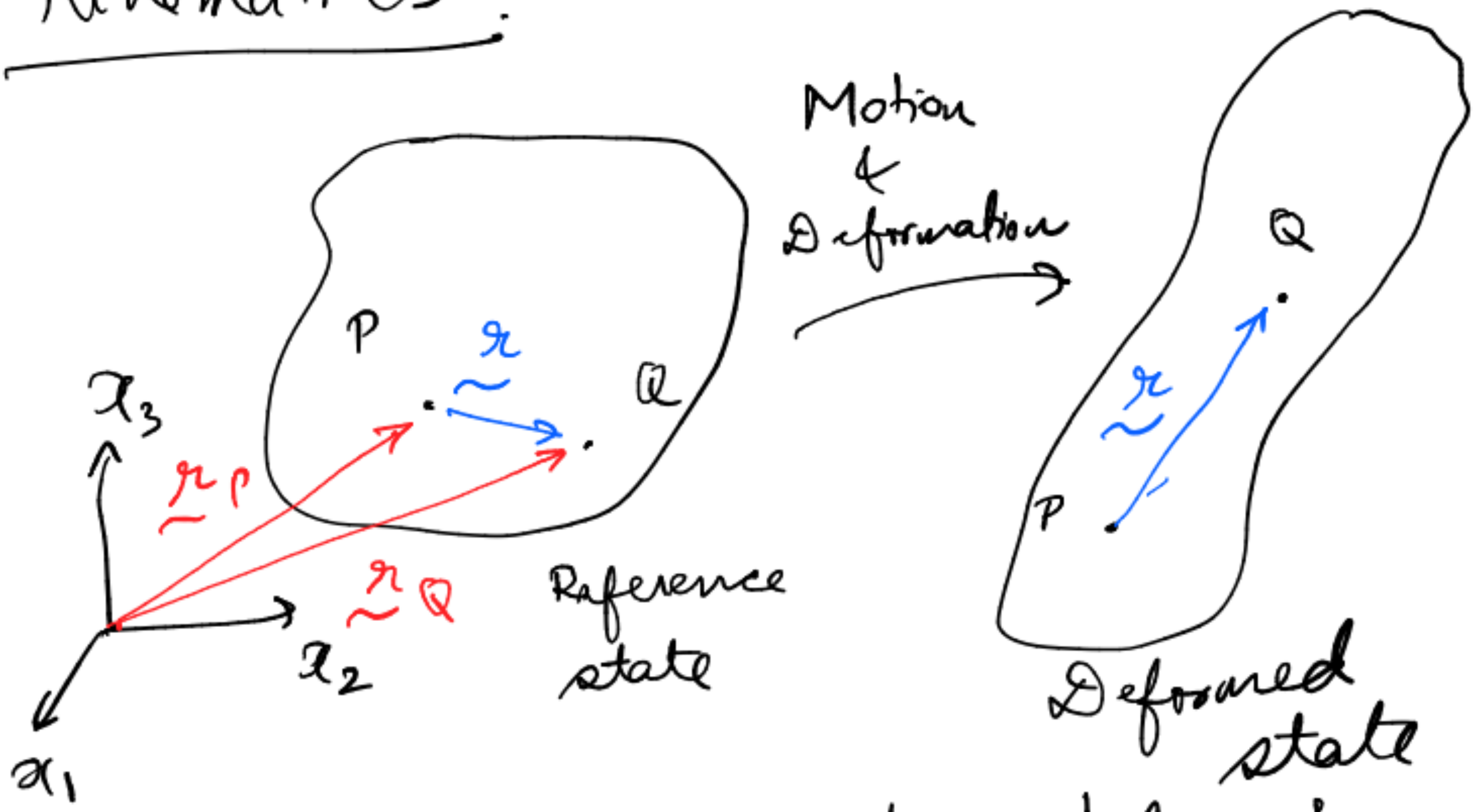
AS5900

Lecture 9

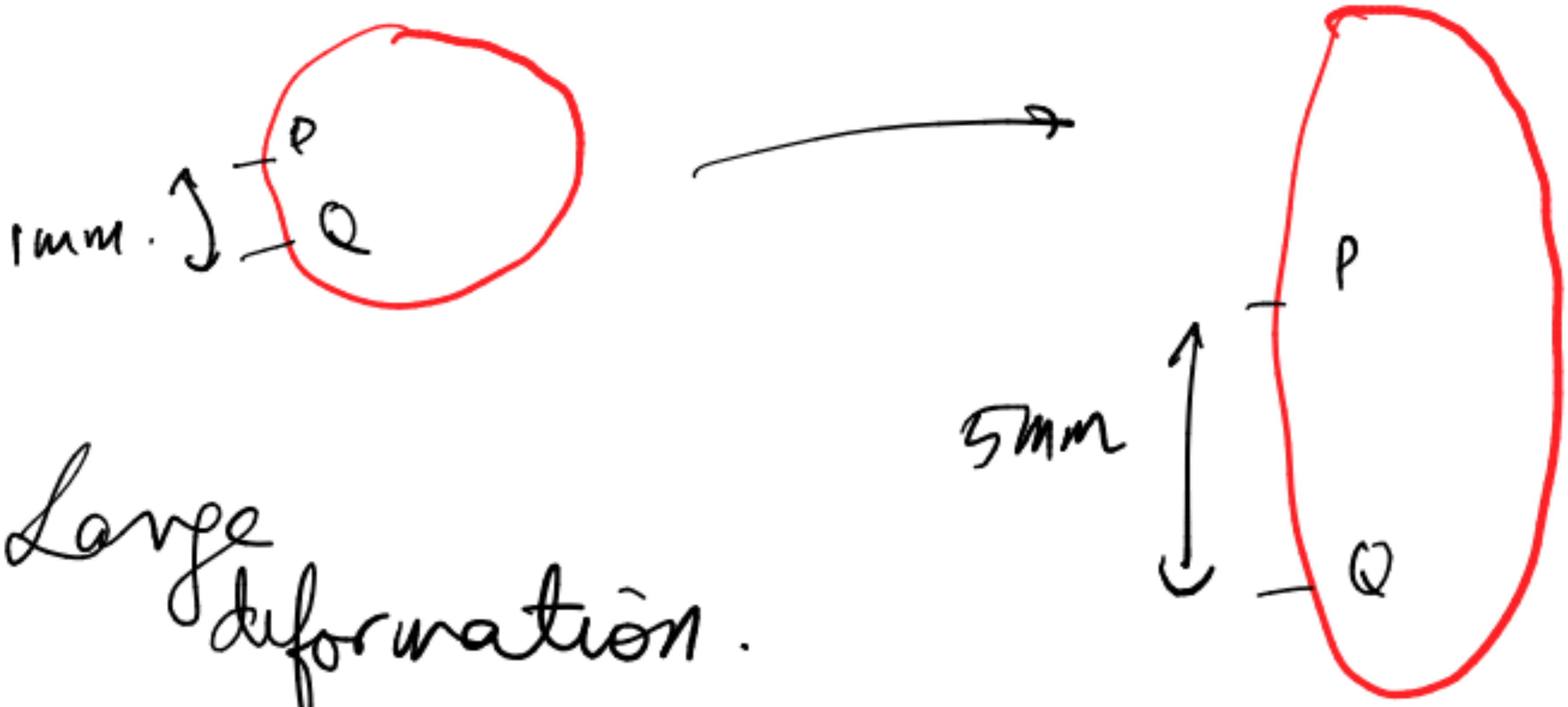
Kinematics:



Motion: Rigid body translation & rotation — $\|\underset{\sim}{r}\| = \|\underset{\sim}{r}'\|$.

Deformation: $\|\underset{\sim}{r}\| \neq \|\underset{\sim}{r}'\|$.

Large deformation: $\|\underset{\sim}{r}'\| - \|\underset{\sim}{r}\|$ is comparable to $\|\underset{\sim}{r}\|$.



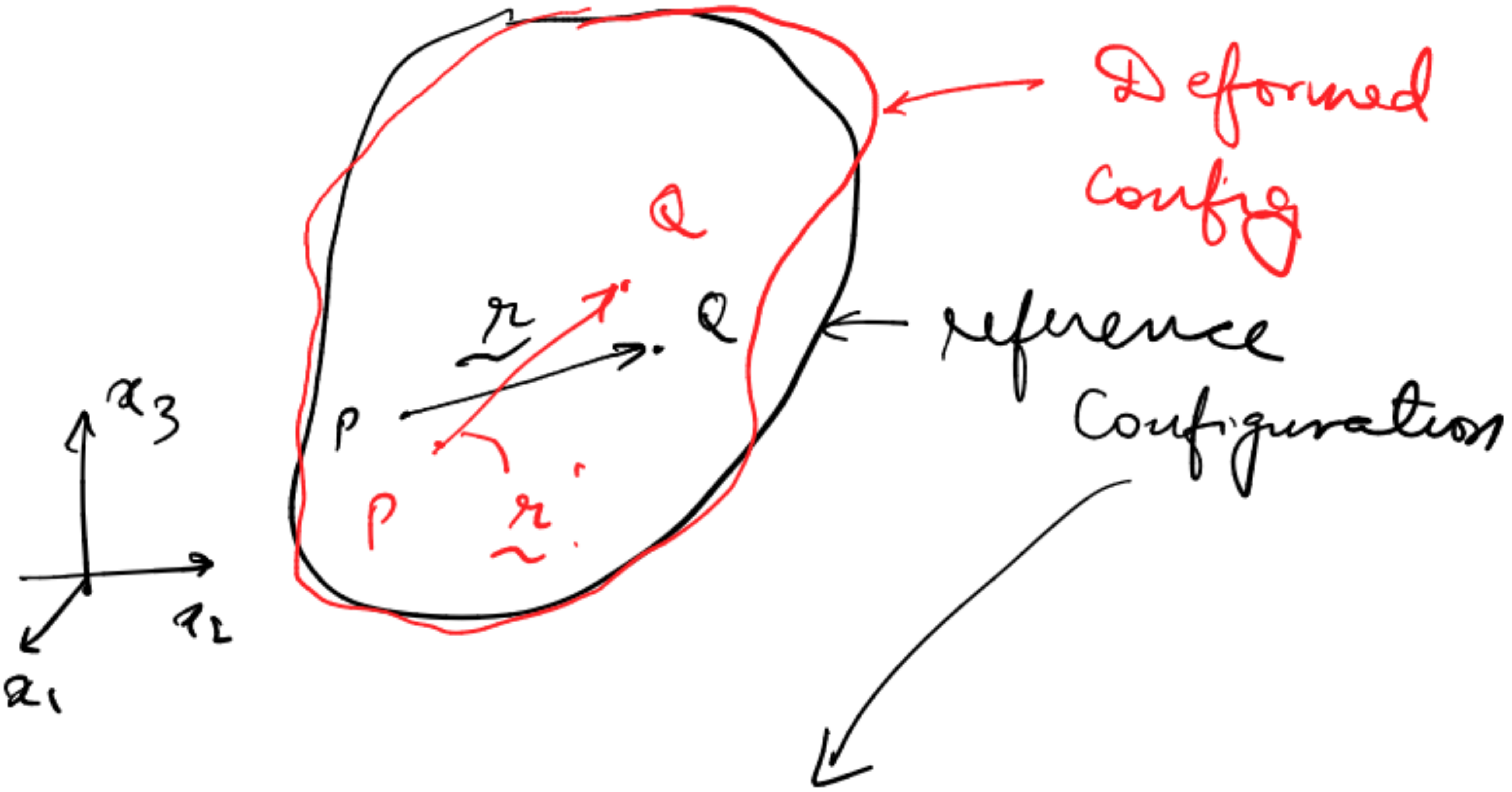
Large deformation.

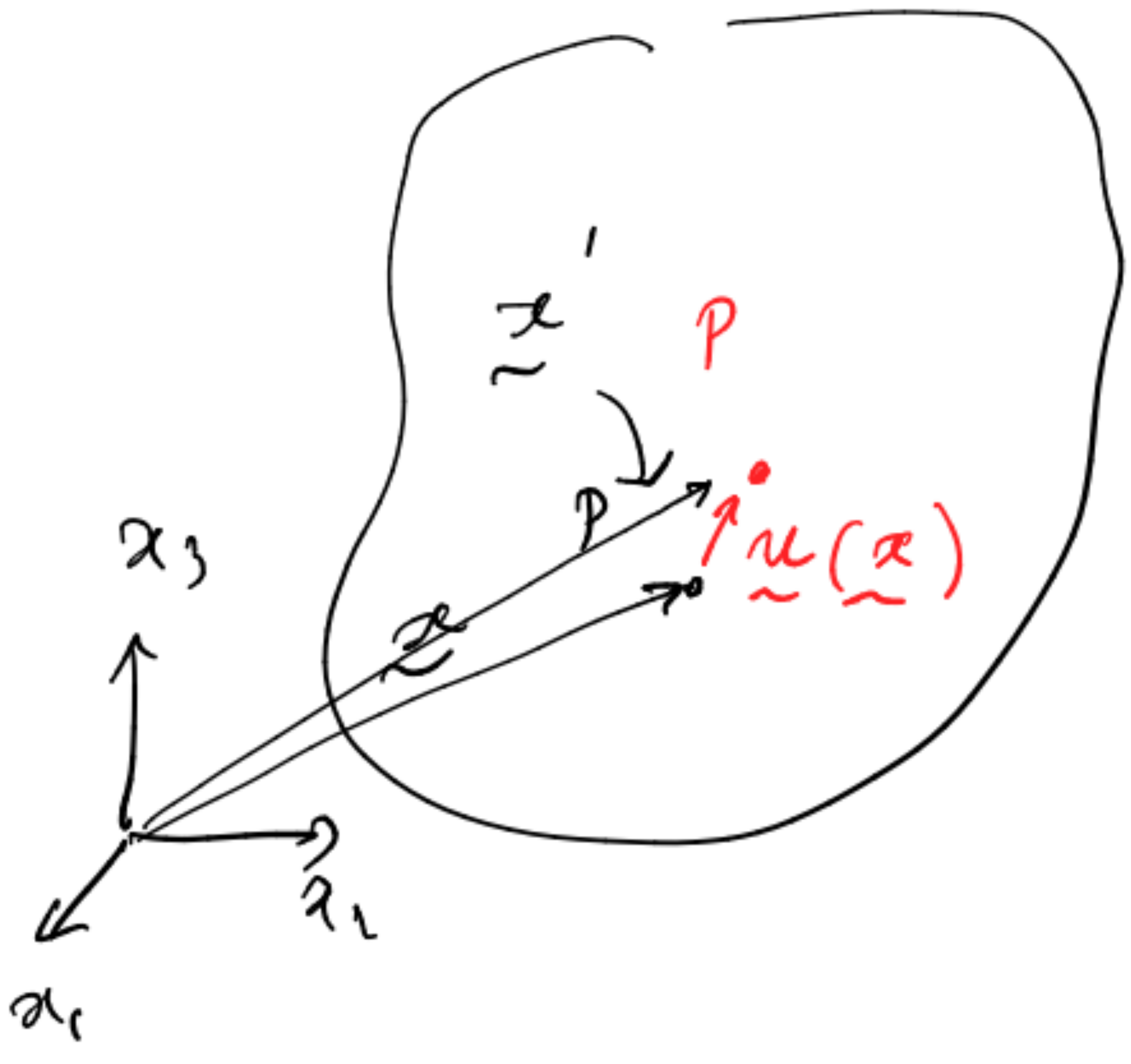
Structural deformations tend to be small.

$$\underbrace{\|\underset{\sim}{r'}\| - \|\underset{\sim}{r}\|}_{\text{change in length.}} \ll \|\underset{\sim}{r}\| \simeq \|\underset{\sim}{r'}\|.$$

The theory of linear elasticity applies only to the case of small deformation.

Small Deformations:





$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{x}' - \underset{\sim}{x}$$

is called the displacement field.

$\underset{\sim}{u}(\underset{\sim}{x})$ is defined $\forall\ \underset{\sim}{x}$ in the body.

$\|\underset{\sim}{u}(\underset{\sim}{x})\| << \|\underset{\sim}{x}\|$ in linear elasticity.

Let

$$\underset{\sim}{u}(\underset{\sim}{x}) = u_i \, \underset{\sim}{e_i}.$$

$$= u_1 \underset{\sim}{e_1} + u_2 \underset{\sim}{e_2} + u_3 \underset{\sim}{e_3}.$$

①

$$\underset{\substack{u_2\\u_3}}{u_1}(\underset{\sim}{x}+\Delta\underset{\sim}{x}) = \underset{\substack{u_2\\u_3}}{u_1}(\underset{\sim}{x}) + \frac{\partial \overset{\substack{u_3\\u_2}}{u_1}}{\partial x_1}\Delta x_1 + \frac{\partial \overset{\substack{u_3\\u_2}}{u_1}}{\partial x_2}\Delta x_2 + \frac{\partial \overset{u_2\;u_3}{u_1}}{\partial x_3}\Delta x_3.$$

+ higher order terms ignored because of small deformation.

The displacement gradient tensor $\nabla \underset{\sim}{u}$ is a 2 tensor whose components in the $x_1 x_2 x_3$ coordinate system:

$$u_{i,j} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

In index notation, write Eq. ① as;

$$u_i(\underset{\sim}{x} + \Delta \underset{\sim}{x}) = u_i(\underset{\sim}{x}) + u_{i,j}\,\Delta x_j \quad ①$$

$$\Rightarrow \underbrace{u_i(\underset{\sim}{x} + \Delta \underset{\sim}{x}) - u_i(\underset{\sim}{x})}_{\text{change in relative position}} = u_{i,j}\,\Delta x_j \quad ②$$

$$\Rightarrow \underset{\sim}{u}(\underset{\sim}{x} + \Delta \underset{\sim}{x}) - \underset{\sim}{u}(\underset{\sim}{x}) = (\nabla \underset{\sim}{u})(\Delta \underset{\sim}{x}).$$

Example:

Let
$$\underset{\sim}{u}^{(x)} = A x^2 y \, \underset{\sim}{e}_1 + B y z \, \underset{\sim}{e}_2 + C x z^3 \, \underset{\sim}{e}_3$$

A, B, C arbitrary constants.

Find $(\nabla \underset{\sim}{u})_{ij}$

$$(\nabla u)_{ij} = u_{i,j} = \begin{pmatrix} 2Axy & Ax^2 & 0 \\ 0 & Bz & By \\ Cz^3 & 0 & 3Cxz^2 \end{pmatrix}.$$

Note: This is not symmetric.

Decompose $(\nabla \underset{\sim}{u})$ into symmetric & skew symmetric parts.

$$A_{ij} = \text{symm}(A_{ij}) + \text{skew}(A_{ij}).$$

$$\text{symm}(A_{ij}) = \frac{A_{ij} + A_{ji}}{2}$$

$$\text{skew}(A_{ij}) = \frac{A_{ij} - A_{ji}}{2}$$

$\text{symm}(A_{ij}) = \frac{A_{ij} + A_{ji}}{2}$ will be symmetric

$$\begin{tikzcd}
\Rightarrow \left(\text{symm}(A_{ij})\right)^T & = & \text{symm}(A_{ij}). \arrow[dd] \\
\left(\frac{A_{ij} + A_{ji}}{2}\right)^T \arrow[u] & & \\
\frac{A_{ij}^T \overset{\parallel}{+} A_{ji}^T}{2} & = & \frac{A_{ji} + A_{ij}}{2}
\end{tikzcd}$$

skew(A_{ij}) satisfies:

$$\begin{tikzcd}
\left(\mathrm{skew}(A_{ij})\right)^T \arrow[r, phantom, "="] & -\left(\mathrm{skew}(A_{ij})\right). \arrow[ddddl] \\
\left(\dfrac{A_{ij}-A_{ji}}{2}\right)^T \arrow[u] \arrow[d, phantom, "\parallel"] & \\
\dfrac{A^T_{ij}-A^T_{ji}}{2} \arrow[d, phantom, "\parallel"] & \\
\dfrac{A_{ji}-A_{ij}}{2} \arrow[d, phantom, "\parallel"] & \\
-\left(\dfrac{A_{ij}-A_{ji}}{2}\right) &
\end{tikzcd}$$

Decompose

$$u_{i,j} = \text{symm}(u_{i,j}) + \text{skew}(u_{i,j})$$

$$= \varepsilon_{ij} + \omega_{ij}$$

$\underset{\sim}{\varepsilon}$ = strain tensor

$\underset{\sim}{\omega}$ = rotation tensor.

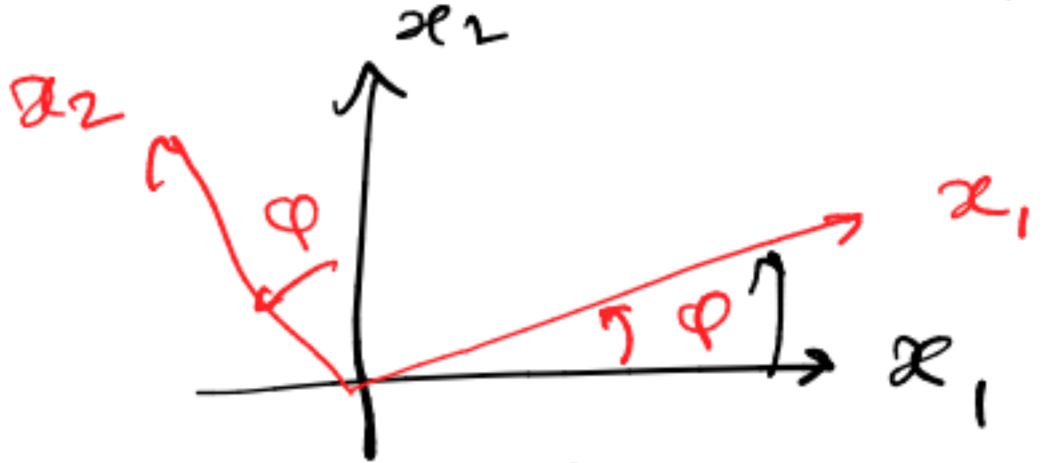
$$\left.\begin{aligned} \varepsilon_{ij} &= \frac{u_{i,j} + u_{j,i}}{2} \\ \omega_{ij} &= \frac{u_{i,j} - u_{j,i}}{2} \end{aligned}\right) \textcircled{3}$$

Subst ③ into ② to get:

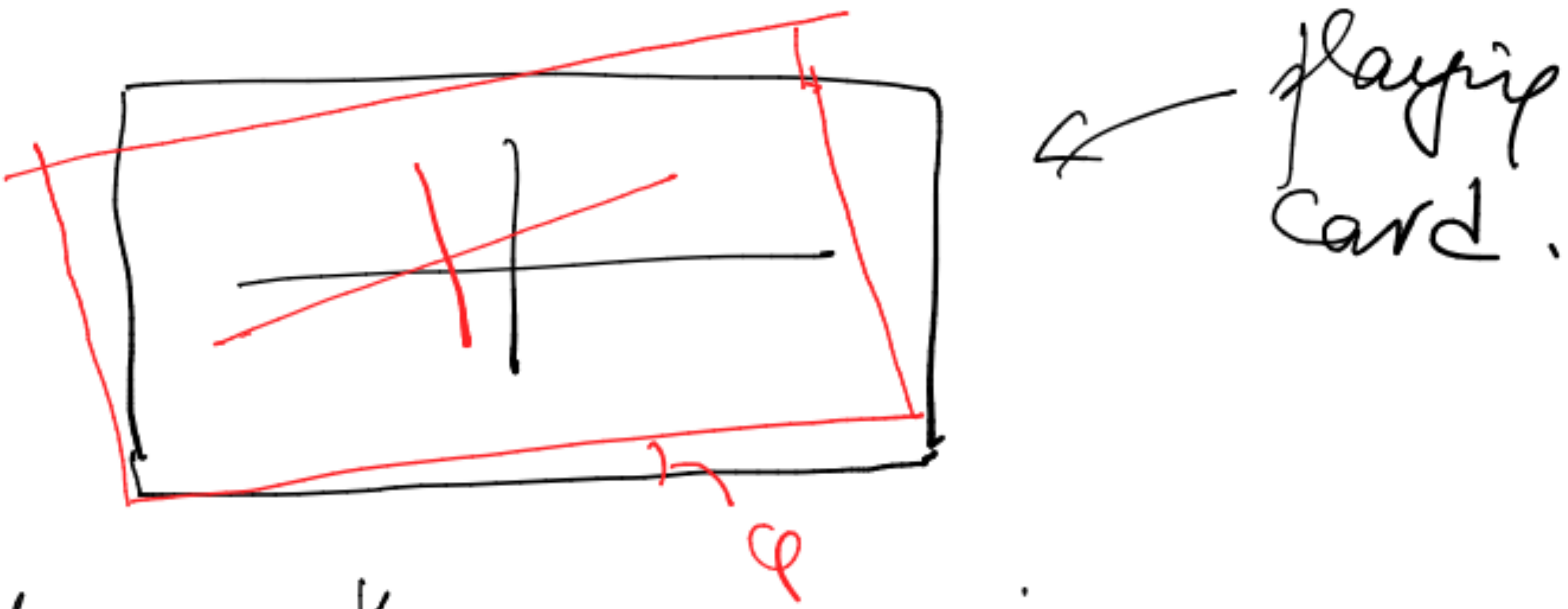
$$u_i(\underset{\sim}{x} + \Delta \underset{\sim}{x}) = \underset{\sim}{u}(\underset{\sim}{x}) + \varepsilon_{ij}\, \Delta x_j + \omega_{ij}\, \Delta x_j$$

$\llcorner$ ④.

Why is ω_{ij} called rotation?



Determine the displacement gradient



Along the x_1-axis

$$u_2 = x_1 \tan\varphi \simeq x_1 \varphi.$$