# A fast algorithm for fracture simulations representing fibre breakage and matrix failure in three-dimensional fibre composites 

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#### Abstract

A linear, periodic, three-dimensional shear-lag model of unidirectionallyreinforced composites that allows for fibre breakage, and matrix failure is proposed. Matrix failure can take the form of matrix splitting or interfacial debonding. A computationally efficient scheme for its solution is developed. This scheme exploits the translation invariance of the elastostatic fields due to failed elements in the periodic cell, and is asymptotically faster than the classical eigensolution-based approach. The new computational scheme is used to illustrate the influence of matrix failure on the elastostatic fields induced by small clusters of fibre breaks in several test problems. Monte Carlo simulations of fracture in model three-dimensional composite specimen with Weibull-distributed fibre segment strengths are also performed. Matrix failure is found to considerably alter fracture development, to weaken the median specimen, and to reduce the variability in composite strength.


Keywords Shear-lag models; Influence function; Fast Fourier transform; Interacting damage; Stress redistribution

## 1 Introduction

Shear-lag models of unidirectional fibre composites loaded uniaxially along the fibre direction assume that fibres carry the applied tension, while matrix elements conduct loads from fibre breaks to nearby fibres by deforming in simple shear (Cox 1952; Hedgepeth 1961; Hedgepeth and Van Dyke 1967). When the fibre, and matrix materials are linear elastic, and the interface between them is perfectly bonded, the elastostatic state in the composite can be computed by weighted superposition of the elastostatic states due to individual fibre breaks (Sastry and Phoenix 1993; Beyerlein et al 1996). Computationally, this approach is much more efficient than full domain solutions using e.g., the finite element method (Xia et al 2002; Mishnaevsky Jr and Dai 2014; Swolfs et al 2013). For this reason, shear-lag

[^0]analyses are the preferred stress analysis tool in fracture simulations of linear fibre composites (Beyerlein and Phoenix 1997).

Because of their utility in fracture simulations, shear-lag models have been extended in various ways. Some of these developments, most relevant to the present work, are now briefly reviewed. The Hedgepeth (1961) and Hedgepeth and Van Dyke (1967) shear-lag models apply to an infinite composite comprised of infinitely many, infinitely long fibres. However, fracture simulations can only be performed on finite simulation cells. Using the Hedgepeth (1961) or Hedgepeth and Van Dyke (1967) model in a finite simulation cell causes load leakage from the simulation cell, i.e., some of the load dropped by every broken fibre goes to unbreakable fibres outside the simulation cell. This spuriously strengthens the model composite represented in the simulation cell.

Addressing this issue, Fukunaga et al (1984) and Landis et al (2000) considered a composite domain periodic in the directions transverse to the fibre direction. In the mathematical formulation of their model, a matrix describes the interactions amongst the fibres in the periodic cell. This interaction matrix is augmented to represent interactions across periodic boundaries. The eigenvalues and eigenvectors of the interaction matrix determine the elastostatic state in the periodic cell with a single fibre break. The computational cost of the eigenvalue problem is $O\left(N^{3}\right)$, where $N$ is the number of fibres in the simulation cell. A further calculation, also costing $O\left(N^{3}\right)$ is required to compute the elastostatic state due to multiple fibre breaks.

Shear-lag models predict that load dropped by broken fibres redistribute preferentially amongst nearby intact fibres. This causes fibre breaks to cluster together in fracture simulations. The transverse periodicity of the shear-lag model of Fukunaga et al (1984) and Landis et al (2000) ensures that clusters of fibre breaks formed at the edges of the simulation cell can continue to extend through the opposite periodic boundary. In other words, clusters are not affected by the transverse cell boundaries. However, while the model specimen of Fukunaga et al (1984) and Landis et al (2000) is infinitely long along the fibre direction, the simulation cell is finite. This implies that the growth of fibre break clusters located near the fibre-wise edges of the simulation cell will be impeded artificially. To overcome this, Mahesh and Phoenix (2004) proposed a simulation cell that is periodic in both the transverse, and fibre directions. Clusters of breaks near any edge of the fully periodic simulation cell continue to grow from the opposite edge of the cell.

In three-dimensional fracture simulations (Ibnabdeljalil and Curtin 1997; Landis et al 2000), composite specimen fail by the catastrophic propagation of a localised cluster of breaks. The critical volume to which the localised cluster must grow before it propagates catastrophically depends, among other things, on the fibre strength variability. For sufficiently small fibre strength variability, the volume encompasses only one fibre break, which suffices to trigger the sequential failure of neighbours (Habeeb and Mahesh 2015). If the fibre strength variability is larger, specimen fracture initiates by the failure of a critical volume comprised of many fibres. On the one hand, in order to be representative of the fracture mechanism in the physical composite, the simulation cell should be much larger than the critical volume of the localised cluster. On the other hand, the solution of the periodic shear-lag model is intrinsically computationally intensive. In order to meet the former requirement subject to the latter limitation, the studies of Ibnabdeljalil and Curtin (1997), and Landis et al (2000) were restricted to the case of relatively
small fibre strength variability. Reducing the $O\left(N^{3}\right)$ computational cost of solving for the elastostatic state in a partly damaged simulation cell using an improved algorithm would lessen the limitation, and allow fracture simulations for larger $N$. It is a major objective of the present work is to propose such an algorithm.

Another approach pursued in the literature to achieve larger $N$ is to restrict the fracture simulations to two-dimensional patches (Mahesh et al 2002). Efficient $O(N \log N)$ algorithms applied to such patches enable fracture simulations in $N>$ $10^{6}$ fibre patches (Mahesh et al 2019), three orders of magnitude larger than those of Ibnabdeljalil and Curtin (1997), and Landis et al (2000). The large $N$ allowed Mahesh et al (2019) to access large fibre strength variability.

Two-dimensional simulation patches extend one characteristic length along the fibre-direction, and fibre breaks are limited to a transverse section located at the mid-point of the characteristic length. The transverse alignment of the fibre breaks enhances stress concentrations due to fibre breaks, and promotes cluster growth. The fibre breaks in the patch are assumed to be mechanically isolated from those in all other patches. Overloads on intact fibres in two-dimensional patches also increases monotonically with the number of fibre breaks, which considerably simplifies load-stepping in the simulation algorithm.

The assumptions underlying the two-dimensional simplification are, however, not realistic. While the load dropped by an isolated fibre break is approximately regained over one characteristic length in a three-dimensional composite, the distance over which load dropped by a cluster of fibre breaks is regained increases with cluster size (Mahesh and Mishra 2018). It is thus not true that fibre breaks, more than one characteristic length apart, are independent. Also, fibre breaks typically form in a staggered fashion. Assuming them to be concentrated in a common transverse plane is unrealistic, and has been shown to underestimate the composite strength (Curtin 2000). Two-dimensional simulations thus do not faithfully capture the pattern of microscopic failure events in physical composites. They are useful, however, to predict a conservative bound on the strength distribution of a three-dimensional composite.

In the works referenced above, fibre breakage is taken to be the only microscopic mechanism of progressive damage. The matrix is assumed to deform in shear, but to remain either perfectly bonded (Landis et al 2000), or to be damaged uniformly to the point of not influencing further load-sharing amongst fibres (Ibnabdeljalil and Curtin 1997). While this is a reasonable assumption in some systems, e.g., carbon-epoxy composites, widespread matrix damage in the form of interfacial debonding, and fibre pull-out is observed e.g., in glass-epoxy composites (Hull and Clyne 1996). These composites exhibit a brush-like fracture surface. Neglecting matrix damage is sometimes justified by invoking the large stiffness contrast between the fibres and the matrix, and noting the smallness of the energy released during matrix failure events. However, recently, Sheikh and Mahesh (2018) have pointed out in the context of hybrid composites that although matrix failure releases little energy, it changes the pattern of load redistribution in the composite, and thereby alters the sequence of further fibre breakage. The latter has a large effect on the fracture energy. A major objective of the present work is to propose a model that accounts for matrix failure.

The objectives of the present work are: (i) To propose a linear, fully periodic, three-dimensional shear-lag model that accounts for fibre breakage, and matrix failure in the form of matrix splitting or interfacial debonding (Sec. 2); (ii) to de-


Fig. 1: A rhombus-shaped periodic patch of $v \times v$ fibers arranged in a hexagonal lattice. The $m-n$ coordinate system is also shown.
termine the influence of fibre breaks, and matrix failures, and to incorporate these influences in an influence superposition technique (Sec. 3), and (iii) to propose a fast Fourier transform-based $O(N \log N)$ algorithm for influence superposition (Sec. 3). The fast algorithm exploits the linearity, and periodicity of the influence fields. Fracture simulations incorporating the fast algorithm are briefly described in Sec. 4, including a novel algorithm to detect specimen fracture. The computational speed-up offered by the present eigensolution-based approach is quantified in Sec. 5, and the nature of load redistribution from broken fibres to intact ones in the presence of matrix failure is analysed. Also, sample fracture simulations with and without matrix failure are presented. These simulations elucidate the role of matrix failure on fracture development.

## 2 Shear-lag model

2.1 Periodic patch and governing equations

The transverse cross-section of the model composite is assumed rhombus-shaped, as shown in Fig. 1, following Mahesh et al (2002). Two edges of the rhombus define the $m$, and $n$ coordinate axes. Fibres are identified by their ( $m, n$ ) coordinates. Following Landis et al (2000), the following periodicity conditions are imposed: fibers $(0, n)$ and $(v-1, n)$ (the left and right edges of the patch) are assumed adjacent for all $n \in\{0,1, \ldots, v-1\}$. Similarly, fibers at the top $(n=v-1)$ and bottom ( $n=0$ ) edges are also considered adjacent. On account of periodicity, only fibre indices modulo $v$ are significant. For indices $m$, and $n$, these are defined as

$$
\begin{align*}
& {[m] }:=m-v\lfloor m / v\rfloor, ~ a n d ~  \tag{1}\\
& {[n]:=n-v\lfloor n / v\rfloor, }
\end{align*}
$$

where $\lfloor\cdot\rfloor$ denotes the largest integer no greater than its argument. Here, and elsewhere, := indicates a definition.

Hedgepeth and Van Dyke (1967) proposed a model for the load distribution in a fibre composite assuming that the fibres are only loaded in tension and the matrix only in shear. Let
$\mathscr{N}_{m n}=\{([m+1], n),(m,[n+1]),([m-1],[n+1]),([m-1], n),(m,[n-1]),([m+1],[n-1])\}$
denote the set of six immediate neighbours of fibre ( $m, n$ ). Assuming each fibre ( $m, n$ ) to receive shear only from the matrix bays connecting it to its six neighbours, they expressed its equilibrium equation along the fibre-wise, or $z$-direction, in terms of its displacement fields $v_{m n}(z)$ as:

$$
\begin{equation*}
E A \frac{d^{2} v_{m n}}{d z^{2}}(z)+\frac{G w}{d} \sum_{\left(m^{\prime}, n^{\prime}\right) \in \mathscr{N}_{m n}}\left(v_{m^{\prime} n^{\prime}}(z)-v_{m n}(z)\right)=0 \tag{3}
\end{equation*}
$$

Here, $E A$ represents the extensional rigidity of a fibre, $G$ the shear modulus of the matrix, $d$ the fibre diameter, and $w$ the inter-fibre spacing. The factor $G w / d$ represents the shear flow in the fibre per unit matrix shear strain. Eq. (3) can alternately be expressed as:

$$
\begin{equation*}
E A \frac{d^{2} v_{m n}}{d z^{2}}(z)+\frac{G w}{d} \sum_{p=0}^{v-1} \sum_{q=0}^{v-1} A_{m n p q} v_{p q}(z)=0 \tag{4}
\end{equation*}
$$

for $m, n \in\{0,1, \ldots, v-1\}$. The interaction between the nearest neighbour fibres is captured through the matrix

$$
A_{\text {mnpq }}= \begin{cases}-6, & \text { if } p=m, \text { and } q=n,  \tag{5}\\ 1, & \text { if } p=[m \pm 1], \text { and } q=n, \\ 1, & \text { if } p=m, \text { and } q=[n \pm 1] \\ 1, & \text { if } p=[m \pm 1], \text { and } q=[n \mp 1] \\ 0, & \text { otherwise. }\end{cases}
$$

The $A_{m n p q}$ matrix is translation invariant (Hedgepeth and Van Dyke 1967; Gupta et al 2017), i.e.,

$$
\begin{equation*}
A_{m n p q}=A_{[m-p],[n-q], 00}, \tag{6}
\end{equation*}
$$

on account of the periodicity of the composite patch.
Hedgepeth and Van Dyke (1967) eliminated the material and geometric constants appearing in Eq. (4) by defining the non-dimensional axial position $\zeta$ as:

$$
\begin{equation*}
\zeta:=z \sqrt{\frac{G w}{E A d}} \tag{7}
\end{equation*}
$$

and the non-dimensional displacement $u_{m n}$ as

$$
\begin{equation*}
u_{m n}:=v_{m n} \sqrt{\frac{E A G w}{d P^{2}}} \tag{8}
\end{equation*}
$$

Here, $P$ denotes the average load per fibre. The appearance of $P$ in the denominator of the right side in Eq. (8) indicates that the non-dimensional displacement, $u_{m n}(z)$,
corresponds to unit average load per fibre. In terms of $\zeta$, and $u_{m n}(\zeta)$, Eq. (4) becomes

$$
\begin{equation*}
\frac{d^{2} u_{m n}}{d \zeta^{2}}(\zeta)+\sum_{p=0}^{v-1} \sum_{q=0}^{v-1} A_{m n p q} u_{p q}(\zeta)=0 \tag{9}
\end{equation*}
$$

Presently, the average load per fibre is assigned a value of unity, i.e.,

$$
\begin{equation*}
\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \frac{d u_{m n}}{d \zeta}(\zeta)=1, \text { for all } \zeta \tag{10}
\end{equation*}
$$

Following Beyerlein et al (1996), the auxiliary displacement field, $\tilde{u}_{m n}(\zeta)$ is defined as

$$
\begin{equation*}
\tilde{u}_{m n}(\zeta)=u_{m n}(\zeta)-\zeta \tag{11}
\end{equation*}
$$

Expressed in terms of $\tilde{u}_{m n}$, Eqs. (9), and (10) become:

$$
\begin{equation*}
\frac{d^{2} \tilde{u}_{m n}}{d \zeta^{2}}(\zeta)+\sum_{p=0}^{v-1} \sum_{q=0}^{v-1} A_{m n p q} \tilde{u}_{p q}(\zeta)=0 \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \frac{d \tilde{u}_{m n}}{d \zeta}(\zeta)=0, \text { for all } \zeta \tag{13}
\end{equation*}
$$

The model composite patch is assumed to extend over $[-L, L]$ along the fibre direction and to be periodic across the boundaries $\zeta= \pm L$, i.e., fibre and matrix elements at $\zeta=\lim _{\varepsilon \rightarrow 0} L-\varepsilon$, and $\zeta=\lim _{\varepsilon \rightarrow 0}-L+\varepsilon$ are assumed adjacent. This requires that any fibre break at $\zeta=L$, is matched by another in the same fibre at $\zeta=-L$, and that if there is no fibre break in fibre $(m, n)$ at $\zeta= \pm L$, the normalised auxiliary displacement $\tilde{u}_{m n}$ obeys (Mahesh and Phoenix 2004)

$$
\begin{equation*}
\tilde{u}_{m n}(\zeta=-L)=\tilde{u}_{m n}(\zeta=L)+S \tag{14}
\end{equation*}
$$

where $S$ is independent of $m$ and $n$. It also requires that the normalised auxiliary traction,

$$
\begin{equation*}
\tilde{\sigma}_{m n}:=\frac{d \tilde{u}_{m n}}{d \zeta} \tag{15}
\end{equation*}
$$

is continuous across the periodic boundary:

$$
\begin{equation*}
\tilde{\sigma}_{m n}(\zeta=-L)=\tilde{\sigma}_{m n}(\zeta=L) \tag{16}
\end{equation*}
$$

for all fibres $(m, n)$ unbroken at $\zeta= \pm L$.


Fig. 2: Longitudinal section along the $n=0$ plane showing the discretisation along the fibre direction. The composite patch is divided into $2 K$ blocks. Each block in turn is divided into $P-1$ segments by $P$ Chebyshev points.

### 2.2 Discretisation of the composite patch

The domain of the periodic composite patch of length $2 L$ is discretised into $2 K$ 'blocks' of equal length, $\Delta=L / K$, indexed as $k=0, k=1, \ldots, k=2 K-1$, as shown in Fig. 2. Periodicity implies that blocks $k=0$ and $k=2 K-1$ abut each other. Paralleling Eq. (1), the periodic index of block $k$ is denoted [ $k$ ], and defined as

$$
\begin{equation*}
[k]:=k-2 K\left\lfloor\frac{k}{2 K}\right\rfloor, \tag{17}
\end{equation*}
$$

so that the neighbours of block $k$ can be expressed simply as [ $k-1$ ], and $[k+1]$.
The block boundary between blocks $[k-1]$ and $[k]$ is located at $\zeta=-L+k \Delta$, for $k \in\{0,1,2, \ldots, 2 K-1\}$. The triplet $(m, n, k)$ identifies a point in the fibre $(m, n)$ that is located at $\zeta=-L+k \Delta$. Field variables associated with this point are identified by the subscript mnk, e.g.,

$$
\begin{equation*}
\tilde{u}_{m n k}:=\tilde{u}_{m n}(\zeta=-L+k \Delta), \text { and } \tilde{\sigma}_{m n k}:=\tilde{\sigma}_{m n}(\zeta=-L+k \Delta) \tag{18}
\end{equation*}
$$

To identify matrix bays, a more elaborate convention is required. Consider the matrix bay in block $k$, between neighbouring fibres $(m, n)$ and $\left(m_{1}, n_{1}\right) \in \mathscr{N}_{m n}$, defined in Eq. (2). Index $i$ is set such that

$$
i:= \begin{cases}0, & \text { if }\left(m_{1}, n_{1}\right)=([m+1], n),  \tag{19}\\ 1, & \text { if }\left(m_{1}, n_{1}\right)=(m,[n+1]), \\ 2, & \text { if }\left(m_{1}, n_{1}\right)=([m-1],[n+1]), \text { and } \\ \text { undefined, } & \text { otherwise }\end{cases}
$$

To avoid double identifying a matrix bay, $i$ is defined only for half of the $\left(m_{1}, n_{1}\right) \in$ $\mathscr{N}_{m n}$. The quadruplet ( $m, n, k, i$ ) uniquely identifies the matrix bay in block $k$ originating from $(m, n)$. All the matrix bays can be identified in this manner, which implies that there are thrice as many matrix bays as fibres in the model patch.

Field variables associated with the matrix bay ( $m, n, k, i$ ) are indicated using subscripts $m n k i$, e.g., $\tilde{\tau}_{m n k i}(\zeta)$, is the normalised shear stress in matrix bay ( $m, n, k, i$ ), defined later in Eq. (52).

### 2.3 Fibre breaks, and matrix failure

Damage in the model composite can take the form of fibre breaks and matrix failures. Fibre breaks are restricted to the boundaries of blocks, while matrix failures are required to span the length of one or more blocks, i.e., extend over $\zeta \in\left(-L+[k] \Delta,-L+\left[k^{\prime}\right] \Delta\right)$, for some integers $k$ and $k^{\prime},[k]<\left[k^{\prime}\right]$. These assumptions are not restrictive because the ideal case of arbitrarily located breaks, and matrix failures is approached by letting $\Delta \downarrow 0$.

Let there be $B$ fibre breaks, located at $\left(m_{b}, n_{b}, k_{b}\right), b \in\{1,2, \ldots, B\}$. The condition of zero traction at fibre break located at $\left(m_{b}, n_{b}, k_{b}\right)$ is

$$
\begin{equation*}
\frac{d u_{m_{b} n_{b} k_{b}}}{d \zeta}=0 \tag{20}
\end{equation*}
$$

or, in terms of the auxiliary displacement $\tilde{u}_{m n}$ of Eq. (11),

$$
\begin{equation*}
\frac{d \tilde{u}_{m_{b} n_{b} k_{b}}}{d \zeta}=-1 \tag{21}
\end{equation*}
$$

Let there be $F$ matrix failures, located at ( $\left.m_{f}, n_{f}, k_{f}, i_{f}\right)$, for $f \in\{1,2, \ldots, F\}$. The model matrix bay ( $m_{f}, n_{f}, k_{f}, i_{f}$ ) is said to be failed if it does not transmit shear between the two fibres flanking it over the axial extent of block $k_{f}$. Debonding of either or both fibre-matrix interfaces flanking the matrix bay (Hull and Clyne 1996), or matrix splitting (Wolla and Goree 1987) are two ways to physically realise matrix failure, as defined. The present definition, however, does not encompass
matrix cracking transverse to the fibre direction, as in ceramic matrix composites (Evans and Zok 1994).

Matrix failures will alter the governing equations of the flanking fibres. Let $\mathscr{I}_{m n k} \subseteq \mathscr{N}_{m n}$ be the set of neighbours of fibre ( $m, n$ ), such that the matrix bays between $(m, n)$ and the elements of $\mathscr{I}_{m n k}$ are intact in block $k$. In the absence of matrix failures, $\mathscr{I}_{m n k}=\mathscr{N}_{m n}$. In the presence of matrix failures, the governing equations, Eq. (3), expressed in terms of the non-dimensional auxiliary displacement fields, $\tilde{u}_{m n}(\zeta)$ become:

$$
\begin{equation*}
\frac{d^{2} \tilde{u}_{m n}}{d \zeta^{2}}(\zeta)+\sum_{\left(m^{\prime}, n^{\prime}\right) \in \mathscr{\mathscr { I }}_{m n k}}\left(\tilde{u}_{m^{\prime} n^{\prime}}(\zeta)-\tilde{u}_{m n}(\zeta)\right)=0 \tag{22}
\end{equation*}
$$

in block $k$. A fibre-interaction matrix, $A_{m n p q}^{\prime}(k)$, constant over each block, and encapsulating the information about all the failed matrix bays can be derived from Eq. (22) for each block $k$ (Sheikh and Mahesh 2018).

Fig. 3 schematically shows the $n=0$ longitudinal section of a composite with 2 fibre breaks and 26 failed matrix bays. For an arbitrary configuration of fibre breaks and matrix failures, it is required to solve the governing equations, Eq. (22), subject to traction free boundary conditions at breaks, Eq. (21) and imposed unit loading, Eq. (13) for the unknown auxiliary displacement fields, $\tilde{u}_{m n}$.

A solution methodology for this problem based on eigenvector expansion was proposed by Sheikh and Mahesh (2018). In this method, $\tilde{u}_{m n}$ within each block is obtained using the eigenvalues and eigenvectors of the $A_{m n p q}^{\prime}(k)$ matrices, together with conditions demanding continuity of displacements and tractions between blocks. However, the eigenvector expansion method is associated with a large asymptotic computational cost of $O\left(v^{6}\right)$ floating point operations. This severely limits the number of fibres and volume of the model composite. The objective of the present work is to propose a computationally much lighter solution methodology, which exploits the structure of the $A_{m n p q}$ matrix, and the computational efficiency of the fast Fourier transform.

## 3 Computationally efficient solution

The present computationally efficient solution algorithm obtains $\tilde{u}_{m n}(\zeta)$ by superposing the elastostatic states corresponding to two unit problems. The two unit problems can be solved, and their solutions superposed efficiently using the fast Fourier transform. An important feature of the present approach, which distinguishes it from that of Sheikh and Mahesh (2018), is that matrix failure is accounted for, not by writing a separate $A_{m n p q}(k)$ for each block $k$, but by treating the matrix as intact throughout and canceling out the shears that the failed matrix bays apply on their flanking fibres. This is done by applying equal and opposite forces on the flanking fibres externally. The determination of these forces is the most computationally challenging aspect. Mathematically, the latter problem has the structure of a saddle point problem (Benzi et al 2005).

$$
(m, n)=(0,0) \quad(m, n)=(1,0) \quad(m, n)=(2,0) \quad(m, n)=(3,0) \quad(m, n)=(v-1,0)
$$

Fig. 3: Longitudinal section along the $n=0$ plane schematically showing a model composite patch with fibre breaks (thick horizontal line segments), and failed matrix bays (hatched regions). Although the model allows matrix failure in threedimensions, for clarity, the damaged state is depicted in one longitudinal plane only.

### 3.1 Governing equations in Fourier space

The two-dimensional discrete Fourier transform (DFT) of $\tilde{u}_{m n}(\zeta)$ is defined as (Briggs and Henson 1995)

$$
\begin{equation*}
\tilde{U}_{r s}(\zeta):=\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \tilde{u}_{m n}(\zeta) \exp \left(-\frac{2 \pi \iota m r}{v}\right) \exp \left(-\frac{2 \pi \iota n s}{v}\right) \tag{23}
\end{equation*}
$$

where $\iota=\sqrt{-1}$. The inverse transform is then

$$
\begin{equation*}
\tilde{u}_{m n}(\zeta)=\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \tilde{U}_{r s}(\zeta) \exp \left(\frac{2 \pi \iota m r}{v}\right) \exp \left(\frac{2 \pi \iota n s}{v}\right) \tag{24}
\end{equation*}
$$

for $j, k \in\{0,1, \ldots, v-1\}$. Gupta et al (2017) have shown that applying Eqs. (23) and (24) to Eq. (12), and utilising the key property given by Eq. (6) yields the governing equation in Fourier space:

$$
\begin{equation*}
\frac{d^{2} \tilde{U}_{j k}}{d \zeta^{2}}(\zeta)+\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} D_{j k r s} \tilde{U}_{r s}(\zeta)=0 \tag{25}
\end{equation*}
$$

for $j, k \in\{0,1, \ldots, v-1\}$, where,

$$
D_{j k r s}= \begin{cases}\sum_{m=0}^{v-1} \sum_{n=0}^{v-1} A_{m n 00} \exp \left(-\frac{2 \pi \iota r m}{\nu}\right) \exp \left(-\frac{2 \pi \iota s n}{\nu}\right), & \text { if } j=r, \text { and } k=s  \tag{26}\\ 0, & \text { otherwise }\end{cases}
$$

In the terminology of Rezghi and Elden (2011), $D_{j k r s}$ is diagonal in the modes $(1,3)$, and $(2,4)$. The diagonal entries of $-D_{j k r s}$ are denoted

$$
\begin{equation*}
\delta_{r s}:=-D_{r s r s} \tag{27}
\end{equation*}
$$

This permits rewriting Eq. (25) as:

$$
\begin{equation*}
\frac{d^{2} \tilde{U}_{r s}}{d \zeta^{2}}(\zeta)-\delta_{r s} \tilde{U}_{r s}(\zeta)=0 \tag{28}
\end{equation*}
$$

for $r, s \in\{0,1, \ldots, v\}$. Eq. (28) represents the uncoupled form of the system of ordinary differential equations, Eq. (25), which can be solved in closed form:

$$
\tilde{U}_{r s}(\zeta)= \begin{cases}c_{r s} \zeta+d_{r s}, & \text { if } \delta_{r s}=0, \text { and }  \tag{29}\\ c_{r s} \exp \left(-\sqrt{\delta_{r s}} \zeta\right)+d_{r s} \exp \left(\sqrt{\delta_{r s}} \zeta\right), & \text { if } \delta_{r s} \neq 0\end{cases}
$$

For the $A_{\text {mnpq }}$ given by Eq. (5), there is exactly one $\delta_{r s}$, which is zero. Without loss of generality, this mode is assigned the indices, $r=s=0$, i.e., $\delta_{00}=0$. Also, for all $r, s \in\{1,2, \ldots, v-1\}, \delta_{r s}>0$, so that the arguments of the exponential functions in Eq. (29) are real (Gupta et al 2017).
$\tilde{U}_{r s}(\zeta)$ must satisfy the condition of zero nett normal traction in any transverse section, as demanded by Eq. (13). Substituting Eq. (24) into Eq. (13) gives:

$$
\begin{equation*}
\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \frac{d \tilde{U}_{r s}}{d \zeta}(\zeta) \exp \left(\frac{2 \pi \iota m r}{v}\right) \exp \left(\frac{2 \pi \iota n s}{v}\right)=0, \text { for all } \zeta \in(-L, L) \tag{30}
\end{equation*}
$$

Switching the order of summation and rearranging yields:

$$
\begin{equation*}
\frac{1}{v^{2}} \sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \frac{d \tilde{U}_{r s}}{d \zeta}(\zeta) \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \exp \left(\frac{2 \pi \iota m r}{v}\right) \exp \left(\frac{2 \pi \iota n s}{v}\right)=0, \text { for all } \zeta \in(-L, L) \tag{31}
\end{equation*}
$$

Now,

$$
\sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \exp \left(\frac{2 \pi \iota m r}{v}\right) \exp \left(\frac{2 \pi \iota n s}{v}\right)= \begin{cases}0, & \text { if } r \neq 0 \text { or } s \neq 0  \tag{32}\\ v^{2}, & \text { if } r=s=0\end{cases}
$$

Eqs. (31), and (32) together imply that $d \tilde{U}_{00} / d \zeta=0$. In other words, $c_{r s}=0$ if $r=s=0$, and Eq. (29), becomes:

$$
\tilde{U}_{r s}(\zeta)= \begin{cases}d_{r s}, & \text { if } r=s=0, \text { and }  \tag{33}\\ c_{r s} \exp \left(-\sqrt{\delta_{r s}} \zeta\right)+d_{r s} \exp \left(\sqrt{\delta_{r s}} \zeta\right), & \text { if } r \neq 0, \text { or } s \neq 0\end{cases}
$$

The Fourier transform of the normalised tractions is then readily obtained by differentiating Eq. (33) as:

$$
\frac{d \tilde{U}_{r s}}{d \zeta}(\zeta)= \begin{cases}0, & \text { if } r=s=0, \text { and }  \tag{34}\\ -c_{r s} \sqrt{\delta_{r s}} \exp \left(-\sqrt{\delta_{r s}} \zeta\right)+d_{r s} \sqrt{\delta_{r s}} \exp \left(\sqrt{\delta_{r s}} \zeta\right), & \text { if } r \neq 0, \text { or } s \neq 0\end{cases}
$$

### 3.2 Fibre breaks

### 3.2.1 Single fibre break

The problem of a single fibre break subject to unit opening displacement was originally formulated and solved by Hedgepeth (1961). A computationally faster solution, based on the fast Fourier transform (FFT) was proposed by Gupta et al (2017). The latter solution assumed an infinitely long simulation cell along the fibre direction. The present solution follows that of Gupta et al (2017), but additionally accounts for the periodicity of the composite patch in the fibre direction, $\zeta$. The treatment of shear stresses due to fibre breaks is also developed here for the first time.

Consider a finite patch extending in the fibre direction over $\zeta \in[-L, L]$, as shown in Fig. 4. Let the fibre $(m, n)=(0,0)$ be broken at $\zeta=0$. The composite is divided into two blocks. Block A extends over $\zeta \in[-L, 0)$, while block B extends over $\zeta \in(0, L]$. Let $\tilde{u}_{m n}^{(A)}(\zeta)$ and $\tilde{u}_{m n}^{(B)}(\zeta)$ denote the auxiliary displacement fields in blocks A and B, respectively. Following Hedgepeth and Van Dyke (1967), let the broken fibre be given a unit opening displacement, i.e.,

$$
\begin{equation*}
\tilde{u}_{00}^{(A)}(\zeta=0)=-\tilde{u}_{00}^{(B)}(\zeta=0)=-1 . \tag{35}
\end{equation*}
$$

Symmetry across the $\zeta=0$ plane dictates that

$$
\begin{equation*}
\tilde{u}_{m n}^{(A)}(\zeta=0)=\tilde{u}_{m n}^{(B)}(\zeta=0)=0, \quad \text { for }(m, n) \neq(0,0) \tag{36}
\end{equation*}
$$

Also, continuity of fibre displacement across the periodic boundary $\zeta= \pm L$, Eq. (14), requires:

$$
\begin{equation*}
\tilde{u}_{m n}^{(A)}(\zeta=-L)=\tilde{u}_{m n}^{(B)}(\zeta=L)+S \tag{37}
\end{equation*}
$$

where $S$ is independent of $m$, and $n$ (Mahesh and Phoenix 2004), and traction continuity across the $\zeta= \pm L$ boundary, Eq. (16), requires:

$$
\begin{equation*}
\frac{d \tilde{u}_{m n}^{(A)}}{d \zeta}(\zeta=-L)=\frac{d \tilde{u}_{m n}^{(B)}}{d \zeta}(\zeta=L) \tag{38}
\end{equation*}
$$

Let the Fourier transforms of $\tilde{u}_{m n}^{(A)}$, and $\tilde{u}_{m n}^{(B)}$, given by Eq. (23), be $\tilde{U}_{r s}^{(A)}(\zeta)$, and $\tilde{U}_{r s}^{(B)}$, respectively. Substituting Eqs. (35) and (36) into Eq. (23) yields:

$$
\begin{equation*}
-\tilde{U}_{r s}^{(A)}(\zeta=0)=\tilde{U}_{r s}^{(B)}(\zeta=0)=-\frac{1}{v^{2}} \tag{39}
\end{equation*}
$$



Fig. 4: Longitudinal section along the $n=0$ plane showing the unit loaded single break.
for $r, s \in\{0,1, \ldots, v-1\}$. Similarly, Fourier transforming Eq. (37) gives

$$
\begin{align*}
\tilde{U}_{r s}^{(A)}(\zeta=-L) & =\tilde{U}_{r s}^{(B)}(\zeta=L)+\frac{S}{v^{2}} \sum_{m=0}^{v-1} \exp \left(-\frac{2 \pi \iota m r}{v}\right) \sum_{n=0}^{v-1} \exp \left(-\frac{2 \pi \iota n s}{v}\right) \\
& = \begin{cases}\tilde{U}_{r s}^{(B)}(\zeta=L)+S, & \text { if }(r, s)=(0,0), \\
\tilde{U}_{r s}^{(B)}(\zeta=L), & \text { for other }(r, s) .\end{cases} \tag{40}
\end{align*}
$$

The second line of Eq. (40) is obtained using

$$
\sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \exp \left(-\frac{2 \pi \iota m r}{v}\right) \exp \left(-\frac{2 \pi \iota n s}{v}\right)= \begin{cases}0, & \text { if } r \neq 0 \text { or } s \neq 0  \tag{41}\\ v^{2}, & \text { if } r=s=0\end{cases}
$$

The Fourier transform of Eq. (38) is simply:

$$
\begin{equation*}
\frac{d \tilde{U}_{r s}^{(A)}}{d \zeta}(\zeta=-L)=\frac{d \tilde{U}_{r s}^{(B)}}{d \zeta}(\zeta=L) \tag{42}
\end{equation*}
$$

Following Eq. (33), let the Fourier transformed displacements in the two blocks be:

$$
\begin{align*}
& \tilde{U}_{r s}^{(A)}(\zeta)= \begin{cases}d_{r s}^{(A)}, & \text { if }(r, s)=(0,0), \\
c_{r s}^{(A)} \exp \left(-\sqrt{\delta_{r s}} \zeta\right)+d_{r s}^{(A)} \exp \left(\sqrt{\delta_{r s}} \zeta\right), & \text { if }(r, s) \neq(0,0),\end{cases}  \tag{43}\\
& \tilde{U}_{r s}^{(B)}(\zeta)= \begin{cases}d_{r s}^{(B)}, & \text { if }(r, s)=(0,0), \\
c_{r s}^{(B)} \exp \left(-\sqrt{\delta_{r s}} \zeta\right)+d_{r s}^{(B)} \exp \left(\sqrt{\delta_{r s}} \zeta\right), & \text { if }(r, s) \neq(0,0)\end{cases}
\end{align*}
$$

Substituting Eq. (43) into Eqs. (39), (40), and (42) yields:

$$
\begin{array}{r}
d_{r s}^{(A)}=-d_{r s}^{(B)}=-\frac{1}{v^{2}}, \text { if }(r, s)=(0,0), \text { and }  \tag{44}\\
c_{r s}^{(A)}+d_{r s}^{(A)}=-c_{r s}^{(B)}-d_{r s}^{(B)}=\frac{1}{v^{2}}, \text { if }(r, s) \neq(0,0) ;
\end{array}
$$

$\begin{aligned} d_{r s}^{(A)} & =d_{r s}^{(B)}+S, \text { if }(r, s)=(0,0), \\ c_{r s}^{(A)} \exp \left(\sqrt{\delta_{r s}} L\right)+d_{r s}^{(A)} \exp \left(-\sqrt{\delta_{r s}} L\right) & =c_{r s}^{(B)} \exp \left(-\sqrt{\delta_{r s}} L\right)+d_{r s}^{(B)} \exp \left(\sqrt{\delta_{r s}} L\right), \text { if }(r, s) \neq(0,0) ;\end{aligned}$
and
$-c_{r s}^{(A)} \exp \left(\sqrt{\delta_{r s}} L\right)+d_{r s}^{(A)} \exp \left(-\sqrt{\delta_{r s}} L\right)=-c_{r s}^{(\boldsymbol{B})} \exp \left(-\sqrt{\delta_{r s}} L\right)+d_{r s}^{(B)} \exp \left(\sqrt{\delta_{r s}} L\right)$, if $(r, s) \neq(0,0)$,
respectively. These equations together imply that

$$
\begin{gather*}
S=-2 / v^{2}  \tag{47}\\
d_{r s}^{(A)}=-1 / v^{2}, \text { and } d_{r s}^{(B)}=1 / v^{2}, \tag{48}
\end{gather*}
$$

if $(r, s)=(0,0)$, and

$$
\begin{align*}
& d_{r s}^{(A)}=-c_{r s}^{(B)}=\frac{\exp \left(\sqrt{\delta_{r s}} L\right)}{2 v^{2} \sinh \left(\sqrt{\delta_{r s}} L\right)}, \\
& d_{r s}^{(B)}=-c_{r s}^{(A)}=\frac{\exp \left(-\sqrt{\delta_{r s}} L\right)}{2 v^{2} \sinh \left(\sqrt{\delta_{r s}} L\right)}, \tag{49}
\end{align*}
$$

if $(r, s) \neq(0,0)$. Substituting Eqs. (48) and (49) into Eq. (34), and performing the inverse transform, Eq. (24), for each $\zeta \in\{-L+k \Delta, k \in\{0,1, \ldots, 2 K-1\}\}$ gives
the normalised tractions at all the block boundaries. The associated asymptotic computational effort of this computation is $O\left(v^{2} \log v\right)$ (Briggs and Henson 1995). The $\tilde{u}_{m n}$ so obtained will obey unit opening displacement at the break, following Eq. (35). Let $\psi$ denote the normalised tractions at the two free boundaries at the break:

$$
\begin{equation*}
\psi:=\frac{d \tilde{u}_{00}^{(A)}}{d \zeta}(\zeta=0)=\frac{d \tilde{u}_{00}^{(B)}}{d \zeta}(\zeta=0) \tag{50}
\end{equation*}
$$

In the sequel, the solution corresponding to unit normalised compressive tractions at the break,

$$
\begin{equation*}
\frac{d \tilde{u}_{00}^{(A)}}{d \zeta}(\zeta=0)=\frac{d \tilde{u}_{00}^{(B)}}{d \zeta}(\zeta=0)=-1 \tag{51}
\end{equation*}
$$

will be needed. Exploiting the linearity of the present problem, the corresponding solution is obtained by scaling $c_{r s}^{(A)}, d_{r s}^{(A)}, c_{r s}^{(B)}$, and $d_{r s}^{(B)}$, obtained in Eqs. (48) and (49) by $-1 / \psi$. The normalised tractions developed in fibre ( $m, n$ ) at the boundary between blocks $k$ and $[k-1]$ due to applied unit compressive tractions, Eq. (51), is denoted $\tilde{\sigma}_{m n k}^{\mathrm{f}}$.

### 3.2.2 Shear tractions, and Chebyshev interpolation

The single fibre break of Sec. 3.2.1 at $\left(m^{\prime}, n^{\prime}, k^{\prime}\right)=(0,0, K)$ also induces shear tractions in all the matrix elements. Let the normalised shear tractions in the matrix bay ( $m, n, k, i$ ) be denoted $\tilde{\tau}_{m n k i}^{\mathrm{f}}(\zeta)$, defined following Eq. (19) as:

$$
\tilde{\tau}_{m n k i}^{\mathrm{f}}(\zeta):= \begin{cases}\tilde{u}_{[m+1], n}(\zeta)-\tilde{u}_{m n}(\zeta), & \text { if } i=0,  \tag{52}\\ \tilde{u}_{m,[n+1]}(\zeta)-\tilde{u}_{m n}(\zeta), & \text { if } i=1, \text { and } \\ \tilde{u}_{[m-1],[n+1]}(\zeta)-\tilde{u}_{m n}(\zeta), & \text { if } i=2,\end{cases}
$$

for $\zeta \in(-L+k \Delta,-L+[k+1] \Delta)$. The functions $\tilde{\tau}_{m n k i}(\zeta)$ are presently approximated by Chebyshev polynomials (Mason and Handscomb 2002).

The Chebyshev polynomials of the first kind are defined recursively over $\hat{\zeta} \in$ $[-1,1]$ as $T_{0}(\hat{\zeta}):=1, T_{1}(\hat{\zeta}):=\hat{\zeta}$, and

$$
\begin{equation*}
T_{p+1}(\hat{\zeta}):=2 \hat{\zeta} T_{p}(\hat{\zeta})-T_{p-1}(\hat{\zeta}), \tag{53}
\end{equation*}
$$

for $p \in\{1,2, \ldots\}$ (Mason and Handscomb 2002). A linear transformation from $\hat{\zeta} \in[-1,1]$ to the domain of the $k$-th block, $\zeta \in[-L+k \Delta,-L+[k+1] \Delta]$, is:

$$
\begin{equation*}
\hat{\zeta}:=\frac{2}{\Delta} \zeta+\frac{2 L-(2 k+1) \Delta}{\Delta} . \tag{54}
\end{equation*}
$$

Over the domain of the $k$-th block, the shear tractions $\tilde{\tau}_{m n k i}^{\mathrm{f}}(\zeta)$ can be approximated by the first $P$ Chebyshev polynomials as:

$$
\begin{equation*}
\tilde{\tau}_{m n k i}^{\mathrm{f}}(\zeta) \approx \sum_{p=0}^{P-1} \tilde{a}_{m n k}^{\mathrm{f}}(i, p) T_{p}\left(\frac{2}{\Delta} \zeta+\frac{2 L-(2 k+1) \Delta}{\Delta}\right) \tag{55}
\end{equation*}
$$

By virtue of Eq. (55), the Chebyshev coefficients $\tilde{a}_{m n k}^{\mathrm{f}}(i, p)$, with $i \in\{0,1,2\}$, and $p \in\{0,1, \ldots, P-1\}$ characterise the normalised shear tractions in the matrix bays. Approximating the shear tractions using Chebyshev polynomials is superior to the
simpler approximation with the monomial basis, $\left\{1, \hat{\zeta}, \hat{\zeta}^{2}, \ldots, \hat{\zeta}^{P-1}\right\}$ (Beyerlein and Phoenix 1996), because the latter is ill-conditioned, and suffers from an instability termed the Runge phenomenon (Mason and Handscomb 2002). A second advantage of the approximation, Eq. (55), is that it converges rapidly (Trefethen 2013). The third, and the most important advantage, associated with the approximation, Eq. (55), is that it can be inverted efficiently. The coefficients $\tilde{a}_{m n k}^{\mathrm{f}}(i, p)$ can be computed using the discrete cosine transform (Mason and Handscomb 2002, Sec. 6.3.3) of $\tilde{\tau}_{m n k i}^{f}\left(\zeta_{p, k}\right)$, where $\zeta_{p, k}$ are the $P$ Chebyshev points in the $k$-th block:

$$
\begin{equation*}
\zeta_{p, k}:=-L+\left([k]+\frac{1}{2}\right) \Delta+\frac{\Delta}{2} \cos \left\{\frac{p \pi}{P-1}\right\} \tag{56}
\end{equation*}
$$

for $p \in\{0,1, \ldots, P-1\}$. The $P=6$ Chebyshev points in block $k=2$ are shown in Fig. 2. Using the discrete cosine transform, the $P$ coefficients $\tilde{a}_{m n k}^{\mathrm{f}}(i, p)$, in all the blocks and fibres, can be obtained using an $O\left(K v^{2} P \log P\right)$ computation, which is linear in the number of elements.

For each $\zeta_{p, k}$, substituting Eqs. (48) and (49) into Eq. (33), and performing the inverse transform using Eq. (24), yields the auxiliary displacements $\tilde{u}_{m^{\prime} n^{\prime}}\left(\zeta_{p, k}\right)$ in all the fibres due to the break at $(m, n, k)=(0,0, K)$. Substituting $\tilde{u}_{m^{\prime} n^{\prime}}\left(\zeta_{p, k}\right)$ into Eq. (52) gives $\tilde{\tau}_{m n k i}^{\mathrm{f}}\left(\zeta_{p, k}\right)$. The Chebyshev coefficients, $a_{m n k}^{\mathrm{f}}(i, p)$ of Eq. (55), can then be obtained using the discrete cosine transform (Mason and Handscomb 2002, Sec. 6.3.3).

### 3.2.3 Normal tractions due to multiple fibre breaks

The influence, $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$, of the fibre break located at arbitrary $\left(m^{\prime}, n^{\prime}, k^{\prime}\right)$ at arbitrary ( $m, n, k$ ), is defined as the normal traction induced at ( $m, n, k$ ) by an isolated break at ( $m^{\prime}, n^{\prime}, k^{\prime}$ ). Here, and elsewhere, unprimed and primed indices of the influence matrix are used to the indicate the influenced, and influencing locations, respectively. It is clear from Sec. 3.2.1 that

$$
\begin{equation*}
\lambda_{m n k ; 00 K}^{\mathrm{f}}=\tilde{\sigma}_{m n k}^{\mathrm{f}} . \tag{57}
\end{equation*}
$$

Periodicity of the patch implies that $\lambda_{m^{\prime} n^{\prime} k^{\prime} ; m n k}^{\mathrm{f}}$ is translation invariant, i.e., it satisfies

$$
\begin{equation*}
\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}=\lambda_{\left[m-m^{\prime}\right]\left[n-n^{\prime}\right]\left[k-k^{\prime}\right] ; 000 .}^{\mathrm{f}} \tag{58}
\end{equation*}
$$

Together, Eqs. (57), and (58) completely specify $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$.
It is recalled from Sec. 2.3 that $B$ fibre breaks are located at ( $m_{b}, n_{b}, k_{b}$ ), for $b \in\{1,2, \ldots, B\}$. Let equal and opposite tractions, $w_{m_{b} n_{b} k_{b}}^{\mathrm{f}}$ be applied to the free surfaces of fibre break $b$ by an external agency. Positive applied tractions $w_{m_{b} n_{b} k_{b}}^{\mathrm{f}}$ are assumed to be compressive, for consistency with Eq. (51), and so that a positive $w_{m_{b} n_{b} k_{b}}^{\mathrm{f}}$ will cause opening displacements at the fibre break. It will prove convenient to associate a $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ with all ( $m^{\prime}, n^{\prime}, k^{\prime}$ ) locations, not only those with a fibre break, and to define it to be zero wherever fibres are not broken, i.e.,

$$
w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}} \begin{cases}:=w_{m_{b} n_{b} k_{b}}^{\mathrm{f}}, & \text { if }\left(m^{\prime}, n^{\prime}, k^{\prime}\right)=\left(m_{b}, n_{b}, k_{b}\right), \text { for some } b, \text { and }  \tag{59}\\ : \equiv 0, & \text { if }\left(m^{\prime}, n^{\prime}, k^{\prime}\right) \neq\left(m_{b}, n_{b} \cdot k_{b}\right), \text { for any } b \in\{1,2, \ldots, B\} .\end{cases}
$$

The normalised traction at ( $m, n, k$ ), due to all the breaks is denoted $\bar{\sigma}_{m n k}^{\mathrm{f}}$, and is given by

$$
\begin{align*}
\bar{\sigma}_{m n k}^{\mathrm{f}} & :=\sum_{b=1}^{B} \lambda_{m n k ; m_{b} n_{b} k_{b}}^{\mathrm{f}} w_{m_{b} n_{b} k_{b}}^{\mathrm{f}} . \\
& =\sum_{m^{\prime}=0}^{v-1} \sum_{n^{\prime}=0}^{v-1} \sum_{k^{\prime}=0}^{2 K-1} \lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}} w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}} . \tag{60}
\end{align*}
$$

The first expression sums only over the fibre breaks. Calculating the stress state at all ( $m, n, k$ ) using this expression entails a computational cost of $O\left(2 K v^{2} B\right)$. The second expression sums of over all the grid locations, whether or not they contain a fibre break. The locations lacking a fibre break do not contribute because of Eq. (59). The second expression requires $2 K v^{2}$ multiplications and additions per ( $m, n, k$ ); for all ( $m, n, k$ ), the computational expense entailed is $\left(2 K v^{2}\right)^{2}$. If $B \ll$ $2 K v^{2}$, i.e., the fibres are sparsely broken, the first expression is much more efficient. However, if $B$ and $2 K v^{2}$ are comparable, there is not much computational gain in using the first expression.

An alternate computationally efficient approach, following Gupta et al (2018), to calculate the normalised tractions at all ( $m, n, k$ ) exists. This hinges on the observation, based on Eq. (58), that $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ is block circulant in the modes $(1,4),(2,5)$, and (3, 6), following the terminology of Rezghi and Elden (2011). It can therefore be diagonalised using the three-dimensional Fourier transform (Briggs and Henson 1995). The diagonal elements are:

$$
\begin{equation*}
\Lambda_{r s t}^{\mathrm{f}}:=\frac{1}{2 K v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \sum_{k=0}^{2 K-1} \lambda_{m n k ; 000}^{\mathrm{f}} \exp \left(-\frac{2 \pi \iota r m}{v}\right) \exp \left(-\frac{2 \pi \iota s n}{v}\right) \exp \left(-\frac{2 \pi \iota t k}{2 K}\right) \tag{61}
\end{equation*}
$$

for $r, s \in\{0,1, \ldots, v-1\}$, and $t \in\{0,1, \ldots, 2 K-1\}$.
Consider the discrete Fourier transform of $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ :

$$
\begin{equation*}
W_{r s t}^{\mathrm{f}}:=\frac{1}{2 K v^{2}} \sum_{m^{\prime}=0}^{v-1} \sum_{n^{\prime}=0}^{v-1} \sum_{k^{\prime}=0}^{2 K-1} w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}} \exp \left(-\frac{2 \pi \iota r m^{\prime}}{v}\right) \exp \left(-\frac{2 \pi \iota s n^{\prime}}{v}\right) \exp \left(-\frac{2 \pi \iota t k^{\prime}}{2 K}\right) \tag{62}
\end{equation*}
$$

$W_{r s t}^{\mathrm{f}}$ represents the components of $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ in the eigenspace of $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$. The diagonal structure of $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ in its eigenspace is exploited to simplify the computation of Eq. (60). Let $\tilde{\Sigma}_{r s t}$ be the three-dimensional discrete Fourier transform of $\tilde{\sigma}_{m n k}$. Then,

$$
\begin{equation*}
\tilde{\Sigma}_{r s t}=\Lambda_{r s t}^{\mathrm{f}} W_{r s t}^{\mathrm{f}} . \tag{63}
\end{equation*}
$$

Inverting $\tilde{\Sigma}_{r s t}$ through Eq. (24) into real space, formally represented as

$$
\begin{equation*}
\bar{\sigma}_{m n k}^{\mathrm{f}}=\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \sum_{t=0}^{2 K-1} \Lambda_{r s t}^{\mathrm{f}} W_{r s t}^{\mathrm{f}} \exp \left(\frac{2 \pi \iota r m}{v}\right) \exp \left(\frac{2 \pi \iota s n}{v}\right) \exp \left(\frac{2 \pi \iota t k}{2 K}\right) \tag{64}
\end{equation*}
$$

yields the normalised tractions in physical space.
For a given $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$, the calculation of $W_{r s t}^{\mathrm{f}}$ through Eq. (62) entails a computational cost of $O\left(2 K v^{2} \log \left(2 K v^{2}\right)\right.$ ) (Briggs and Henson 1995); the products of

Eq. (63) $2 K v^{2}$ multiplications, and the Fourier inversion of Eq. (64) a further cost of $O\left(2 K v^{2} \log \left(2 K v^{2}\right)\right)$ (Briggs and Henson 1995). This implies that the overall computational effort of obtaining the normalised tractions is $O\left(2 K v^{2} \log \left(2 K v^{2}\right)\right)$, much smaller than the cost of directly evaluating Eq. (60), except when $B$ is very small.

### 3.2.4 Shear tractions due to multiple fibre breaks

The shear traction, $\tilde{\tau}_{m n k i}(\zeta)$, for $\zeta \in(-L+k \Delta,-L+[k+1] \Delta)$, induced by the single fibre break of Sec. 3.2.1 at $\left(m^{\prime}, n^{\prime}, k^{\prime}\right)=(0,0, K)$ in all the matrix bays is characterised by the Chebyshev coefficients, $a_{m n k}(i, p)$, as detailed in Sec. 3.2.2. 3P influence matrices, corresponding to $i \in\{0,1,2\}$, and $p \in\{0,1, \ldots, P-1\}$ are required to specify the influence of fibre breaks on matrix bays. The ( $i, p$ )-th influence matrix, $\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}(i, p)$, is defined as the $p$-th Chebyshev coefficient of the shear traction induced in the matrix bay ( $m, n, k, i$ ) due to a break located at ( $m^{\prime}, n^{\prime}, k^{\prime}$ ). The fibre break of Sec. 3.2.1 is located at $(m, n, k)=(0,0, K)$. From Sec. 3.2.2,

$$
\begin{equation*}
\mu_{m n k ; 00 K}^{\mathrm{f}}(i, p)=a_{m n k}^{\mathrm{f}}(i, p) . \tag{65}
\end{equation*}
$$

On account of periodicity in the model patch, $\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ is also block circulant in the modes $(1,4),(2,5)$, and $(3,6)$ :

$$
\begin{equation*}
\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}(i, p)=\mu_{\left[m-m^{\prime}\right]\left[n-n^{\prime}\right]\left[k-k^{\prime}\right] ; 000}^{\mathrm{f}}(i, p), \tag{66}
\end{equation*}
$$

paralleling Eq. (58). Eqs. (65) and (66) together specify all the elements of the influence matrices, $\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}(i, p)$. The $p$-th Chebyshev coefficient of the shear stress induced in the mnki-th matrix bay can be expressed in analogy with Eq. (60) as:

$$
\begin{align*}
\bar{a}_{m n k}^{\mathrm{f}}(i, p) & :=\sum_{b=1}^{B} \mu_{m n k ; m_{b} n_{b} k_{b}}^{\mathrm{f}}(i, p) w_{m_{b} n_{b} k_{b}}^{\mathrm{f}} . \\
& =\sum_{m^{\prime}=0}^{v-1} \sum_{n^{\prime}=0}^{v-1} \sum_{k^{\prime}=0}^{2 K-1} \mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}(i, p) w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}} . \tag{67}
\end{align*}
$$

Performing the calculations in Fourier space is much more efficient, as noted in Sec. 3.2.3. Accordingly, let $M_{r s t}^{\mathrm{f}}(i, p)$ denote the three dimensional Fourier transform
$M_{r s t}^{\mathrm{f}}(i, p):=\frac{1}{2 K v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \sum_{k=0}^{2 K-1} \mu_{m n k ; 000}^{\mathrm{f}}(i, p) \exp \left(-\frac{2 \pi \iota r m}{v}\right) \exp \left(-\frac{2 \pi \iota s n}{v}\right) \exp \left(-\frac{2 \pi \iota t k}{2 K}\right)$,
for $r, s \in\{0,1, \ldots, v-1\}$, and $t \in\{0,1, \ldots, 2 K-1\}$. As defined in Eqs. (59) and (62), $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}$ and $W_{r s t}^{\mathrm{f}}$ represent the weights of the fibre breaks, and its Fourier transform, respectively. Let $\bar{a}_{m n k}^{\mathrm{f}}(i, p)$ denote the $p$-th Chebyshev coefficient of the shear traction induced in the matrix bay ( $m, n, k, i$ ) by all the fibre breaks in the composite patch. The development spanning Eqs. (60)-(64) concerning the calculation of the normalised tractions in the fibres due to an arbitrary collection of breaks carries over to the present case too. The counterpart of Eq. (64) is:

$$
\begin{equation*}
\bar{a}_{m n k}^{\mathrm{f}}(i, p)=\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \sum_{t=0}^{2 K-1} M_{r s t}^{\mathrm{f}}(i, p) W_{r s t}^{\mathrm{f}} \exp \left(\frac{2 \pi \iota r m}{v}\right) \exp \left(\frac{2 \pi \iota s n}{v}\right) \exp \left(\frac{2 \pi \iota t k}{2 K}\right) . \tag{69}
\end{equation*}
$$

The shear traction in the matrix bay ( $m, n, k, i$ ) due to all the fibre breaks, denoted $\bar{\tau}_{m n k i}^{\mathrm{f}}(\zeta)$, can then be obtained by substituting Eq. (69) into Eq. (55):

$$
\begin{equation*}
\bar{\tau}_{m n k i}^{\mathrm{f}}(\zeta) \approx \sum_{p=0}^{P-1} \bar{a}_{m n k}^{\mathrm{f}}(i, p) T_{p}\left(\frac{2}{\Delta} \zeta+\frac{2 L-(2 k+1) \Delta}{\Delta}\right) \tag{70}
\end{equation*}
$$

### 3.3 Matrix failure

Just like fibre breaks, matrix failure can also induce normal tractions in the fibres, and shear tractions in the matrix bays. However, in the absence of fibre breaks, the matrix does not experience shear loading. In other words, an isolated matrix failure has no influence on the elastostatic fields in the fibres. For this reason, in the present treatment, the problem of equal and opposite distributed forces applied to neighbouring fibres in an undamaged composite patch is considered. In Sec. 3.4, the solution to this problem will be used to cancel out the shear tractions exerted by failed matrix bays in the flanking fibres, and thereby mimic the mechanical effect of matrix failure.

### 3.3.1 Point force pair

Suppose all the fibres and matrix bays in the periodic composite patch are intact. Let normalised forces $\pm 1$ be applied at $\zeta=0$ in fibre $(m, n)=\left(m_{1}, n_{1}\right)$, and $(m, n)=$ $\left(m_{2}, n_{2}\right)$, respectively. The domain of the composite is the same as in Sec. 3.2.1. The domain is divided into two blocks, now named ( $C$ ) and ( $D$ ). Fig. 5 shows a longitudinal section of such a composite for the case that $\left(m_{1}, n_{1}\right)=(0,0)$, and $\left(m_{2}, n_{2}\right)=(1,0)$. The imposition of the forces $\pm 1$ at $\zeta=0$ in fibres $(m, n)=\left(m_{1}, n_{1}\right)$, and $(m, n)=\left(m_{2}, n_{2}\right)$, implies

$$
\frac{d \tilde{u}_{m n}^{(C)}}{d \zeta}(\zeta=0)= \begin{cases}\frac{d \tilde{u}_{m n}^{(D)}}{d(L)}  \tag{71}\\ d=0)-1, & \text { if }(m, n)=\left(m_{1}, n_{1}\right) \\ \frac{d \tilde{u}_{n n}^{(D)}}{d \zeta}(\zeta=0)+1, & \text { if }(m, n)=\left(m_{2}, n_{2}\right), \text { and } \\ \frac{d \tilde{u}_{m n}^{(D n}}{d \zeta}(\zeta=0), & \text { otherwise }\end{cases}
$$

Also, displacements must be continuous across the block interface at $\zeta=0$ :

$$
\begin{equation*}
\tilde{u}_{m n}^{(C)}(\zeta=0)=\tilde{u}_{m n}^{(D)}(\zeta=0) \tag{72}
\end{equation*}
$$

Substituting Eq. (71) into Eq. (23) yields:

$$
\frac{d \tilde{U}_{r s}^{(C)}}{d \zeta}(\zeta=0)=\frac{d \tilde{U}_{r s}^{(D)}}{d \zeta}(\zeta=0)-\frac{1}{v^{2}}\left\{\exp \left(-\frac{2 \pi \iota m_{1} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{1} s}{v}\right)-\exp \left(-\frac{2 \pi \iota m_{2} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{2} s}{v}\right)\right\}
$$

for all $r, s \in\{0,1, \ldots, v-1\}$, while Eq. (72) transforms to

$$
\begin{equation*}
\tilde{U}_{r s}^{(C)}(\zeta=0)=\tilde{U}_{r s}^{(D)}(\zeta=0) \tag{74}
\end{equation*}
$$

Substituting Eq. (43) into Eq. (74) yields:

$$
\begin{align*}
d_{r s}^{(C)} & =d_{r s}^{(D)}, \text { if }(r, s)=(0,0), \\
c_{r s}^{(C)}+d_{r s}^{(C)} & =c_{r s}^{(C)}+d_{r s}^{(D)}, \text { if }(r, s) \neq(0,0) . \tag{75}
\end{align*}
$$



Fig. 5: Longitudinal section along the $n=0$ plane showing the forces exerted on a pair of neighbouring fibres, to shear the in-between matrix bay.

Similarly, substituting Eq. (43) into Eq. (71) yields:

$$
\begin{equation*}
-c_{r s}^{(C)}+d_{r s}^{(C)}=-c_{r s}^{(D)}+d_{r s}^{(D)}-\frac{1}{v^{2} \sqrt{\delta_{r s}}}\left\{\exp \left(-\frac{2 \pi \iota m_{1} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{1} s}{v}\right)-\exp \left(-\frac{2 \pi \iota m_{2} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{2} s}{v}\right)\right\} \tag{76}
\end{equation*}
$$

for $(r, s) \neq(0,0)$. Equations (45) and (46), which represent the continuity of displacements and strains at the periodic boundary $\zeta= \pm L$ are also valid presently.

These equations together imply that $s=0$,

$$
\begin{equation*}
d_{r s}^{(C)}=d_{r s}^{(D)}=0, \tag{77}
\end{equation*}
$$

if $(r, s)=(0,0)$, and
$c_{r s}^{(C)}=d_{r s}^{(D)}=\frac{1}{4 v^{2} \sqrt{\delta_{r s}}} \frac{\exp \left(\sqrt{\delta_{r s}} L\right)}{\sinh \left(\sqrt{\delta_{r s}} L\right)}\left\{\exp \left(-\frac{2 \pi \iota m_{1} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{1} s}{v}\right)-\exp \left(-\frac{2 \pi \iota m_{2} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{2} s}{v}\right)\right\}$, and
$c_{r s}^{(D)}=d_{r s}^{(C)}=\frac{1}{4 v^{2} \sqrt{\delta_{r s}}} \frac{\exp \left(-\sqrt{\delta_{r s}} L\right)}{\sinh \left(\sqrt{\delta_{r s}} L\right)}\left\{\exp \left(-\frac{2 \pi \iota m_{1} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{1} s}{v}\right)-\exp \left(-\frac{2 \pi \iota m_{2} r}{v}\right) \exp \left(-\frac{2 \pi \iota n_{2} s}{v}\right)\right\}$,
for $(r, s) \neq(0,0)$.

### 3.3.2 The distributed force pair

It is recalled from Eq. (53) that $T_{p^{\prime}}(\hat{\zeta})$ denotes the $p^{\prime}$-th order Chebyshev polynomial over $\hat{\zeta} \in[-1,1]$. Suppose equal and opposite distributed forces of intensity, $\pm T_{p^{\prime}}((2 \zeta-\Delta) / \Delta)$ are applied by an external agency on the fibres ( $m_{1}, n_{1}$ ) and ( $m_{2}, n_{2}$ ) over the interval $\zeta \in[0, \Delta]$, i.e., block $K$. Also, attention is restricted to the case that $\left(m_{1}, n_{1}\right)$ and ( $m_{2}, n_{2}$ ) represent the flanking fibres of a matrix bay, $\left(0,0, K, i^{\prime}\right)$, $i^{\prime} \in\{0,1,2\}$, as identified by Eq. (19). Fig. 6 shows $\pm T_{2}(2 \zeta / \Delta+(2 L-(2 K+1) \Delta) / \Delta)$ imposed on the fibres $\left(m_{1}, n_{1}\right)=(0,0)$, and $\left(m_{2}, n_{2}\right)=(1,0)$.

Regarding the distributed force pairs as an assembly of infinitesimal point force pairs, $\pm T_{p^{\prime}}\left(\left(2 \zeta_{0}-\Delta\right) / \Delta\right) d \zeta_{0}$, the displacements $\tilde{U}_{r s}$ in Fourier space due to the distributed force pair, $\pm T_{p^{\prime}}(\hat{\zeta})$, can be obtained by superposing the displacements obtained in Eqs. (33), (77), and (78). This yields: $\tilde{U}_{00}(\zeta)=0$, and

for $r \neq 0$, or $s \neq 0$. Eq. (79) can be evaluated efficiently by pre-calculating and storing the formulae for the indefinite integrals

$$
\begin{align*}
& I_{p^{\prime}, r s}^{+}=\frac{\Delta}{2} \int T_{p^{\prime}}\left(\hat{\zeta}_{0}\right) \exp \left(\frac{\Delta}{2} \sqrt{\delta_{r s}} \hat{\zeta}_{0}\right) d \hat{\zeta}_{0}, \text { and } \\
& I_{p^{\prime}, r s}^{-}=\frac{\Delta}{2} \int T_{p^{\prime}}\left(\hat{\zeta}_{0}\right) \exp \left(-\frac{\Delta}{2} \sqrt{\delta_{r s}} \hat{\zeta}_{0}\right) d \hat{\zeta}_{0} \tag{80}
\end{align*}
$$



Fig. 6: Longitudinal section along the $n=0$ plane showing the distributed forces exerted on a pair of neighbouring fibres, so as to shear the in-between matrix bay.
for $p^{\prime} \in\{0,1, \ldots, P\}$, and $r, s \in\{0,1, \ldots, v-1\}$, using a computer algebra package such as Maxima (2014). In terms of these functions, Eq. (79) can be written as:

$$
\tilde{U}_{r s}(\zeta)=\left\{\begin{array}{ll}
\left\{\left.c_{r s}^{(D)} \exp \left(-\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{+}\right|_{-1} ^{1}+\left.d_{r s}^{(D)} \exp \left(\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{-}\right|_{-1} ^{1}\right\}, & \text { if } \zeta \geq \Delta, \\
\left\{\left.c_{r s}^{(C)} \exp \left(-\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{+}\right|_{-1} ^{1}+\left.d_{r s}^{(C)} \exp \left(\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{-}\right|_{-1} ^{1}\right\}, & \text { if } \zeta \leq 0, \text { and } \\
\left\{\left.c_{r s}^{(D)} \exp \left(-\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{+}\right|_{-1} ^{\zeta-\Delta / 2}+d_{r s}^{(D)} \exp \left(\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{-}| |_{-1}^{\zeta-\Delta / 2}\right\}
\end{array}\right\}, ~ \begin{array}{ll}
\left(\left.\zeta c_{r s}^{(C)} \exp \left(-\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{+}\right|_{\zeta-\Delta / 2} ^{1}+\left.d_{r s}^{(C)} \exp \left(\sqrt{\delta_{r s}}\left(\zeta-\frac{\Delta}{2}\right)\right) I_{p^{\prime}, r s}^{-}\right|_{\zeta-\Delta / 2} ^{1}\right\}, & \text { if } 0<\zeta \leq \Delta,
\end{array}
$$

for $r \neq 0$, or $s \neq 0$, where

$$
\begin{equation*}
\left.I_{p^{\prime}, r s}^{ \pm}\right|_{a} ^{b}=I_{p^{\prime}, r s}^{ \pm}(b)-I_{p^{\prime}, r s}^{ \pm}(a) \tag{82}
\end{equation*}
$$

It is straightforward to obtain the strains in Fourier space, $d \tilde{U}_{r s} / d \zeta$, by differentiating Eq. (81), taking care to employ Leibnitz's rule to determine $d\left(\left.I_{p^{\prime}, r s}^{ \pm}\right|_{-1} ^{\zeta}\right) / d \zeta$, and $d\left(\left.I_{p^{\prime}, r s}^{ \pm}\right|_{\zeta} ^{1}\right) / d \zeta$, where applicable.

Fourier inverting this expression formally using Eq. (24) yields the normal tractions at all the block boundaries, denoted $\tilde{\sigma}_{m n k}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$. These calculations are somewhat tedious, but elementary, and are omitted for brevity. Similarly, the shear tractions induced by the present distributed forces can also be computed substituting the present $\tilde{u}_{m n}(\zeta)$ into Eq. (52). The resulting shear tractions are denoted $\tilde{\tau}_{m n k i}^{\mathrm{m}}\left(\zeta ; i^{\prime}, p^{\prime}\right)$. The Chebyshev coefficients of $\tilde{\tau}_{m n k i}^{\mathrm{m}}\left(\zeta ; i^{\prime}, p^{\prime}\right)$, following Eq. (55) are denoted $a_{m n k}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$. The computations of these coefficients for the present problem of distributed fibre loading exactly parallels the development in Sec. 3.2.2 for the case of fibre breaks.

### 3.3.3 Normal tractions due to multiple distributed loads

The influence, $\left.\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}} i^{\prime}, p^{\prime}\right)$, of the distributed loads $\pm T_{p^{\prime}}(2 \zeta / \Delta+(2 L-(2 K+$ 1) $\Delta$ ) $/ \Delta$ ) applied to a pair of neighbouring fibres flanking the matrix bay $\left(m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}\right)$ at ( $m, n, k$ ) is defined as the auxiliary strain induced by this loading at ( $m, n, k$ ). From Sec. 3.3.2,

$$
\begin{equation*}
\lambda_{m n k ; 00 K}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)=\tilde{\sigma}_{m n k}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) . \tag{83}
\end{equation*}
$$

As in Eq. (58), periodicity of the patch implies that $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$ is translation invariant, i.e., it satisfies:

$$
\begin{equation*}
\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}\left(i^{\prime}, p^{\prime}\right)=\lambda_{\left[m-m^{\prime}\right]\left[n-n^{\prime}\right]\left[k-k^{\prime}\right] ; 000}^{m}\left(i^{\prime}, p^{\prime}\right), \tag{84}
\end{equation*}
$$

and is therefore diagonalisable through through three-dimensional Fourier transformation. Let $\Lambda_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$ be the diagonalised form of $\lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i ;, p^{\prime}\right)$, obtained through Fourier transformation:
$\Lambda_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right):=\frac{1}{2 K v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \sum_{k=0}^{2 K-1} \lambda_{m n k ; 000}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) \exp \left(-\frac{2 \pi \iota r m}{v}\right) \exp \left(-\frac{2 \pi \iota s n}{v}\right) \exp \left(-\frac{2 \pi \iota t k}{2 K}\right)$,
for $r, s \in\{0,1, \ldots, v-1\}$, and $t \in\{0,1, \ldots, 2 K-1\}$.
Let external tractions $\pm w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) T_{p^{\prime}}(2 \zeta / \Delta+(2 L-(2 K+1) \Delta) / \Delta)$ be applied to the fibers flanking matrix bay ( $m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}$ ). Traction is applied only to the fibres flanking a failed matrix bay. For convenience, however, $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$ is defined to be zero for intact matrix bays ( $m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}$ ), and for all $p$ :

$$
w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) \begin{cases}:=w_{m_{f} n_{f} k_{f}}^{\mathrm{m}}\left(i_{f}, p^{\prime}\right) & \text { if }\left(m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}\right)=\left(m_{f}, n_{f}, k_{f}, i_{f}\right), \text { for some } f, \text { and }  \tag{86}\\ : \equiv 0, & \text { if }\left(m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}\right) \neq\left(m_{f}, n_{f}, k_{f}, i_{f}\right), \text { for any } f .\end{cases}
$$

Paralleling Eq. (60), the normal traction at fibre location ( $m, n, k$ ) induced by all the distributed tractions, is

$$
\begin{align*}
\bar{\sigma}_{m n k}^{\mathrm{m}} & :=\sum_{p^{\prime}=0}^{P-1} \sum_{f=1}^{F} \lambda_{m n k ; m_{f} n_{f} k_{f}}^{\mathrm{m}}\left(i_{f}, p^{\prime}\right) w_{m_{f} n_{f} k_{f}}^{\mathrm{m}}\left(i_{f}, p^{\prime}\right), \\
& =\sum_{i^{\prime}=1}^{3} \sum_{p^{\prime}=0}^{P-1} \sum_{m^{\prime}=0}^{v-1} \sum_{n^{\prime}=0}^{v-1} \sum_{k^{\prime}=0}^{2 K-1} \lambda_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right), \tag{87}
\end{align*}
$$

which is expensive to compute directly, for the reasons noted below Eq. (60).
Instead, let $W_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$ be the three-dimensional Fourier transform of $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$, keeping $i^{\prime}$, and $p^{\prime}$ fixed. Then, paralleling Eq. (69), the traction, $\bar{\sigma}_{m n k}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$, induced by all the applied distributed tractions is
$\bar{\sigma}_{m n k}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right):=\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \sum_{t=0}^{2 K-1} \Lambda_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) W_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) \exp \left(\frac{2 \pi \iota r m}{v}\right) \exp \left(\frac{2 \pi \iota s n}{v}\right) \exp \left(\frac{2 \pi \iota t k}{2 K}\right)$.
Eq. (88) must be calculated for each $\left(i^{\prime}, p^{\prime}\right), i^{\prime} \in\{0,1,2\}$, and $p^{\prime} \in\{0,1, \ldots, P-1\}$, and their results superposed in real space to obtain the auxiliary normal tractions:

$$
\begin{equation*}
\bar{\sigma}_{m n k}^{\mathrm{m}}:=\sum_{i^{\prime}=0}^{2} \sum_{p^{\prime}=0}^{P-1} \bar{\sigma}_{m n k}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) . \tag{89}
\end{equation*}
$$

### 3.3.4 Shear tractions due to multiple distributed loadings

The influence, $\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$, of the distributed loads $\pm T_{p^{\prime}}(2 \zeta / \Delta+(2 L-(2 K+$ 1) $\Delta$ )/ $\Delta$ ) applied to a pair of neighbouring fibres flanking the matrix bay ( $m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}$ ) on the matrix bay ( $m, n, k, i$ ) is the $p$-th Chebyshev component of the normalised shear traction induced in the latter. For $\left(m^{\prime}, n^{\prime}, k^{\prime}, i^{\prime}\right)=\left(0,0, K, i^{\prime}\right)$, from Sec. 3.3.2,

$$
\begin{equation*}
\mu_{m n k ; 00 K}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)=a_{m n k}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right) . \tag{90}
\end{equation*}
$$

As in Eq. (58), periodicity of the patch implies that $\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$ is translation invariant for fixed $i, p, i^{\prime}$, and $p^{\prime}$, i.e., it satisfies:

$$
\begin{equation*}
\mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}\left(i, p ; i^{\prime}, p^{\prime}\right)=\mu_{\left[m-m^{\prime}\right]\left[n-n^{\prime}\right]\left[k-k^{\prime}\right] ; 000}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right) \tag{91}
\end{equation*}
$$

Let $\bar{a}_{m n k}\left(i, p ; i^{\prime}, p^{\prime}\right)$ be the $p$-th Chebyshev coefficient of the shear traction induced in matrix bay $m n k i$ due to the imposition of tractions $\pm w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) T_{p^{\prime}}(2 \zeta / \Delta+$ $(2 L-(2 K+1) \Delta) / \Delta)$ in the fibres flanking the matrix bays $m^{\prime} n^{\prime} k^{\prime} i^{\prime}$. Then,

$$
\begin{align*}
\bar{a}_{m n k}\left(i, p ; i^{\prime}, p^{\prime}\right) & :=\sum_{f=1}^{F} \mu_{m n k ; m_{b} n_{b} k_{b}}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right) w_{m_{b} n_{b} k_{b}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) \\
& =\sum_{m^{\prime}=0}^{v-1} \sum_{n^{\prime}=0}^{v-1} \sum_{k^{\prime}=0}^{2 K-1} \mu_{m n k ; m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}\left(i, p ; i^{\prime}, p^{\prime}\right) w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{f}}\left(i^{\prime}, p^{\prime}\right) \tag{92}
\end{align*}
$$

The block circulant structure of $\mu_{m n k ; 000}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$ makes it diagonalisable through through three-dimensional Fourier transformation, paralleling Eq. (61).

Let $M_{r s t}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$ be the diagonalised form of $\mu_{m n k ; 000}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$. Let the weights, $w_{m^{\prime} n^{\prime} k^{\prime}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$ and their Fourier transforms, $W_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$, of the distributed forces applied to neighbouring fibres be as specified in Sec. 3.3.3. Then, the $p$-th Chebyshev coefficients, $\bar{a}_{m n k}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right)$, of the shear tractions due to all the distributed forces for fixed $i^{\prime}$, and $p^{\prime}$ in all the matrix bays are obtained using:
$\bar{a}_{m n k}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right):=\sum_{r=0}^{v-1} \sum_{s=0}^{v-1} \sum_{t=0}^{2 K-1} M_{r s t}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right) W_{r s t}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right) \exp \left(\frac{2 \pi \iota r m}{v}\right) \exp \left(\frac{2 \pi \iota s n}{v}\right) \exp \left(\frac{2 \pi \iota t k}{2 K}\right)$.
Further, paralleling Eq. (89),

$$
\begin{equation*}
\bar{a}_{m n k}^{\mathrm{m}}(i, p):=\sum_{i^{\prime}=0}^{2} \sum_{p^{\prime}=0}^{P-1} \bar{a}_{m n k}^{\mathrm{m}}\left(i, p ; i^{\prime}, p^{\prime}\right) \tag{94}
\end{equation*}
$$

### 3.4 Zero traction conditions

For given fibre weights, $w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}$, and matrix weights $w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right)$, Secs. 3.2, and 3.3 present a method to efficiently compute the normal and shear tractions in all the fibres and matrix bays, respectively. However, $w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}$ and $w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right)$ are not known. They must be determined by demanding zero normal and shear tractions at the sites of fibre breaks, and matrix failure, respectively (Hedgepeth 1961).

The zero normal traction boundary condition at fibre breaks is given by Eq. (21). Substituting Eqs. (60) and (87) into Eq. (21) yields a linear equation in the weights $w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}$, and $w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i^{\prime}, p^{\prime}\right)$.

$$
\begin{align*}
& \sum_{b^{\prime}=1}^{B} \lambda_{m_{b^{\prime \prime}} n_{b^{\prime \prime}} k_{b^{\prime \prime}} ; m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}} w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}+ \\
& \sum_{p^{\prime}=0}^{P-1} \sum_{f^{\prime}=1}^{F} \lambda_{m_{b^{\prime \prime}} n_{b^{\prime \prime}} k_{b^{\prime \prime}} ; m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right) w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right)=-1, \tag{95}
\end{align*}
$$

for $b^{\prime \prime} \in\{1,2, \ldots, B\}$.
Consider next the problem of enforcing zero shear traction in failed matrix bays $\left(m_{f^{\prime \prime}}, n_{f^{\prime \prime}}, k_{f^{\prime \prime}}, i_{f^{\prime \prime}}\right)$, while recalling that the analysis of Secs. 3.2, and 3.3 assumed intact matrix bays. The external tractions imposed on the fibres, $w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{f}}$ and $w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}\left(i_{f^{\prime}}, p^{\prime}\right)$, induce shear tractions characterised by the Chebyshev coefficients $\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}^{\mathrm{f}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)$, given by Eq. (67), and $\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)$, given by Eq. (92). By way of reaction, the matrix bays apply opposite shear tractions on their flanking fibres, characterised by the Chebyshev coefficients $-\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)$ and $-\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}^{\mathrm{m}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)$.

As stated in Sec. 2.3, the mechanical effect of the failure of the matrix bay ( $m_{f^{\prime \prime}}, n_{f^{\prime \prime}}, k_{f^{\prime \prime}}, i_{f^{\prime \prime}}$ ) is that it exerts no tractions on its flanking fibres. This is achieved in the model composite patch with intact matrix bays by imposing external tractions on the flanking fibres. These externally imposed tractions must exactly cancel
the tractions that the flanking fibres experience from failed matrix bays. Mathematically, this condition can be expressed as:

$$
\begin{equation*}
w_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)=\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}^{\mathrm{f}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)+\bar{a}_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right), \tag{96}
\end{equation*}
$$

for all $p^{\prime \prime} \in\{0,1, \ldots, P-1\}$, and all the failed matrix bays ( $m_{f^{\prime \prime}}, n_{f^{\prime \prime}}, k_{f^{\prime \prime}}, i_{f^{\prime \prime}}$ ). It is important to note that Eq. (96) does not demand zero shear strain in the failed matrix bays, i.e., the displacement of the fibres flanking a failed matrix bay will, in general, be unequal. Substituting Eqs. (67), (92), and (89) into Eq. (96) yields:

$$
\begin{align*}
& w_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime}}}^{\mathrm{m}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right)=\sum_{b^{\prime}=1}^{B} \mu_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime} ; m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right) w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}+} \\
& \sum_{p^{\prime}=1}^{P-1} \sum_{f^{\prime}=1}^{F} \mu_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime} ; m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}\left(i_{f^{\prime \prime}}, p^{\prime \prime} ; i_{f^{\prime},}, p^{\prime}\right) w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right),} . \tag{97}
\end{align*}
$$

for each matrix failure, $f^{\prime \prime} \in\{1,2, \ldots, F\}$, and Chebyshev interpolation order, $p^{\prime \prime} \in\{0,1, \ldots, P-1\}$. Let

Eq. (97) can then be rewritten as:

$$
\begin{aligned}
& \sum_{b^{\prime}=1}^{B} \mu_{m_{f^{\prime \prime}} n_{f^{\prime \prime}} k_{f^{\prime \prime} ; m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}\left(i_{f^{\prime \prime}}, p^{\prime \prime}\right) w_{m_{b^{\prime}} n_{b^{\prime}} k_{b^{\prime}}}^{\mathrm{f}}+}^{+}
\end{aligned}
$$

$$
\begin{align*}
& w_{m_{f^{\prime}} n_{f^{\prime}} k_{f^{\prime}}}^{\mathrm{m}}\left(i_{f^{\prime}}, p^{\prime}\right)=0 . \tag{99}
\end{align*}
$$

### 3.5 Iterative solution

Dropping the indices for clarity, and denoting a column vector all of whose elements are $a$ as $\{a\}$, Eqs. (95) and (99) can be written as:

$$
\left[\begin{array}{cc}
{[W]_{B \times B}} & {[X]_{B \times P F}}  \tag{100}\\
{[Y]_{P F \times B}[Z]_{P F \times P F}}
\end{array}\right]\binom{\left\{w^{\mathrm{f}}\right\}_{B \times 1}}{\left\{w^{\mathrm{m}}\right\}_{P F \times 1}}=\binom{\{-1\}_{B \times 1}}{\{0\}_{P F \times 1}} .
$$

Here, the elements of $[W],[X],[Y]$, and $[Z]$ are the coefficients $\lambda^{\mathrm{f}}, \lambda^{\mathrm{m}}, \mu^{\mathrm{f}}$, and $\mu^{\mathrm{m}}-I^{\mathrm{m}}$, respectively. Eq. (100) is a linear system of the saddle point type, which appears in a variety of applications (Benzi et al 2005). The matrix $[W]_{B \times B}$, which represents the interactions between fibre breaks, is known to be symmetric and negative definite (Gupta et al 2018). [Z] $]_{P F \times P F}$ represents the influence between failed matrix bays; this is symmetric but indefinite. The matrix $[X]_{B \times P F}$ represents normal tractions induced by matrix failures, and the matrix $[Y]_{P F \times B}$ represents
shear tractions induced by fibre breaks. $[X] \neq[Y]^{T}$. This makes the coefficient matrix in Eq. (100) unsymmetrical. It is also indefinite.

Direct solution has a computational complexity of $O\left((B+P F)^{3}\right)$, which becomes prohibitive for large $B$ and/or $F$ (Golub and Van Loan 2012). An iterative method, based on Krylov subspace methods is therefore preferred (Greenbaum 1997), especially since the matrix products $[W]\left\{w^{\mathrm{f}}\right\},[X]\left\{w^{\mathrm{m}}\right\},[Y]\left\{w^{\mathrm{f}}\right\}$, and $[Z]\left\{w^{\mathrm{m}}\right\}$ can be computed efficiently, following Secs. 3.2.3, 3.2.4, 3.3.3, and 3.3.4. A robust method to solve Eq. (100) for unsymmetric indefinite systems is the pre-conditioned GMRES method (Saad and Schultz 1986). This method, however, requires a good pre-conditioner that will tightly cluster the eigenvalues of the coefficient matrix in Eq. (100) in the Argand plane (Greenbaum 1997; Saad 2003). Excellent preconditioners are known for some problems, which have special structure (Bai 2006). Most dramatically, if $Z=0$ in Eq. (100), Murphy et al (2000) have proposed a preconditioner that should result in GMRES converging in three, or fewer iterations. However, to the knowledge of the author, the coefficient matrix in Eq. (100) does not conform to the special structure of any of the cases examined in the literature. Therefore, a simple preconditioner is not readily available. It should also be noted that attempts to solve Eq. (100) using GMRES without pre-conditioning fail, as convergence is excruciatingly slow.

In the present work, Uzawa iterations (Uzawa 1958; Elman and Golub 1994), with Anderson acceleration (Anderson 1965; Ho et al 2017) is used to solve Eq. (100) iteratively. Let $t \in\{0,1,2, \ldots\}$ be an iteration counter for Uzawa's iteration. Starting with the initial iterate $(t=0)$ :

$$
\begin{equation*}
\binom{w_{B \times 1}^{\mathrm{f}}}{w_{P F \times 1}^{\mathrm{m}}}_{0}:=\binom{\{1\}_{B \times 1}}{\{0\}_{P F \times 1}}, \tag{101}
\end{equation*}
$$

successive iterates are obtained as (Elman and Golub 1994)

$$
\begin{align*}
-[W]\left\{w^{\mathrm{f}}\right\}_{t+1} & =-\{-1\}+[X]\left\{w^{\mathrm{m}}\right\}_{t}, \text { and }  \tag{102}\\
\left\{w^{\mathrm{m}}\right\}_{t+1} & =\left\{w^{\mathrm{m}}\right\}_{t}+\omega\left([Y]\left\{w^{\mathrm{f}}\right\}_{t+1}+[Z]\left\{w^{\mathrm{m}}\right\}_{t}\right) . \tag{103}
\end{align*}
$$

Exploiting the symmetry and positive-definiteness of $-[W]$, Eq. (102) is solved using the unpreconditioned conjugate gradient method (Greenbaum 1997). The relaxation parameter $\omega$ in Eq. (103) is chosen to be unity. It is reiterated that all the matrix vector products are computed following the efficient Fourier transform based formulae given in Secs. 3.2.3, 3.2.4, 3.3.3, and 3.3.4.

Uzawa's method uses up much less computer memory than GMRES, and does not require sophisticated pre-conditioners, but it converges slowly. Convergence is greatly accelerated using the Anderson acceleration scheme, implemented as detailed by Ho et al (2017). In this method, a small number $M$ of Uzawa iterates are stored, along with their residues. $M=10$, presently. Iterate $t+1$ is generated by solving a least squares problem involving all the $M$ stored iterates and residuals. Details may be found in Ho et al (2017).

## 4 Fracture simulations

The model shear-lag specimen of Sec. 2 is subjected to fracture simulations following the algorithm detailed in e.g., Landis et al (2000). Deviations from this
algorithm are noted below. The most important deviation is that matrix failures are also generated in the course of the simulations, and the overloads due to composite damage are computed using the fast procedure developed in Sec. 3. Briefly, the fracture simulation begins with assigning random strengths to the fibre segments drawn from a specified probability distribution. Presently, the strengths of the fibre segments extending $\Delta / 2$ to either side of locations ( $m, n, k$ ) are taken to be Weibull (1952) distributed:

$$
\begin{equation*}
F(\sigma)=\operatorname{Pr}\{\Sigma \leq \sigma\}=1-\exp \left(-\Delta \sigma^{\rho}\right), \tag{104}
\end{equation*}
$$

where $\rho$ is termed the Weibull modulus. In the first step, the average load per fibre, $P$ (see Eq. (8)), is assigned the strength of the weakest fibre segment, so that it breaks. Fibre breaks are restricted to the locations ( $m, n, k$ ), consistent with the assumption made in Sec. 2.3. Matrix bays, on the other hand, are not assigned strengths. Instead, breakage of a fibre at ( $m, n, k$ ) is assumed to trigger the failure of all the matrix bays abutting it, viz., $(m, n, k, 1),(m, n, k, 2),(m, n, k, 3)$, ([m-1], $n, k, 1),(m,[n-1], k, 2)$, and $([m+1],[n-1], k, 3)$, in block $k$, and matrix bays $(m, n,[k-1], 1),(m, n,[k-1], 2),(m, n,[k-1], 3),([m-1], n,[k-1], 1),(m,[n-1],[k-1], 2)$, and $([m+1],[n-1],[k-1], 3)$ in block $[k-1]$. Simulations in which matrix bays are assumed not to fail are also performed. Comparing the fracture development predicted by these two cases elucidates the role of matrix failure below. The two cases will henceforth be identified by the phrases 'with matrix failure', and 'without matrix failure', respectively.

In the second step, the overload due to the first break at all the other fibre segments is computed. If the tensile stress in any fibre element exceeds its strength, that element is then failed. When no more failures occur, the average load per fibre, $P$, is increased so that one more fibre break forms. The overloads are recomputed, and damage events caused by the overloads are accumulated. This process is repeated in subsequent steps until the model specimen fractures. It must be noticed that the average load per fibre increases monotonically over the fracture simulation, i.e., the simulation mimics a load-controlled test. In this respect too, the present algorithm deviates from the displacement controlled simulations of Landis et al (2000).

The normalised macroscopic strain, $\bar{\epsilon}$ at each step of the simulation is defined as:

$$
\begin{align*}
\bar{\epsilon} & =P\left(\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \frac{u_{m n}(\zeta=L)-u_{m n}(\zeta=-L)}{2 L}\right) \\
& =P\left(1+\frac{1}{v^{2}} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \frac{\tilde{u}_{m n}(\zeta=L)-\tilde{u}_{m n}(\zeta=-L)}{2 L}\right)  \tag{105}\\
& =P\left(1+\frac{2}{v^{2}} \frac{1}{\psi} \frac{1}{2 L} \sum_{m=0}^{v-1} \sum_{n=0}^{v-1} \sum_{k=0}^{2 K-1} w_{m n k}^{\mathrm{f}}\right)
\end{align*}
$$

Here, the second step follows from the first using Eq. (11), and the third step from the second step using Eqs. (37), (47), and (50). $w_{m n k}^{\mathrm{f}}$, defined in Eq. (59), represents the weights of the fibre breaks. Eq. (105) expresses the macroscopic strain as a superposition of contributions from the elasticity of the fibres, and the opening displacements of the breaks.


Fig. 7: Longitudinal section along the $n=0$ plane schematically showing a simulation cell with fibre breaks (thick horizontal line segments), and failed matrix bays (hatched regions). The information about fibre breaks and matrix failures is captured using the graph, shown superimposed. Open circles depict vertices, and lines connecting vertices depict edges of the graph. Each $\Delta$-segment of the fibre has an associated vertex. Neighbouring vertices in the same fibre are connected by an edge if there is no intervening fibre break. Vertices located in the same block, in adjacent fibres, are connected by an edge if the intervening matrix bay is intact. Although the model graph is three-dimensional, for clarity only a two-dimensional longitudinal section is shown.

Landis et al (2000) terminated their simulation when the displacement-controlled loading curve exhibited a significant load drop. This approach is not suitable for the present load-controlled algorithm. Therefore, an explicit termination condition to algorithmically identify a fractured computer specimen is needed. If fibre breaks were the only type of microscopic damage in the model specimen, then specimen fracture is easily identified with the failure of all the fibres in a transverse plane. In this case, the transverse plane containing the broken fibres represents the smooth macroscopic crack. If, as in the present case, matrix failure is also a microscopic damage mechanism, the crack that causes specimen fracture need not lie in a single transverse plane. The fracture surface may comprise of fibre breaks in a number of transverse planes, connected together by failed matrix bays. Algorithmically identifying the formation of the macroscopic crack in this case is not trivial. A graph based algorithm is presently proposed for this purpose.

The graph based algorithm maintains the connectivities in the simulation cell using a graph $G=(V, E)$, where $V$ represents the set of vertices, and $E$ represents the set of edges. Each fibre segment within a $\Delta$-block is associated with a vertex of the graph. Vertex $v_{m n k}$ is identified with the fibre segment in fibre $(m, n)$ included within block $k \in\{0,1, \ldots, 2 K-1\}$. Vertices are connected to each other by edges. Two vertices in adjacent bundles, but in the same fibre, $v_{m n k}$ and $v_{m n[k+1]}$ are connected by an edge, if there is no fibre break at the common block boundary. Each vertex $v_{m n k}$ is also connected by edges horizontally to the vertices $v_{m^{\prime} n^{\prime} k}$, for $\left(m^{\prime}, n^{\prime}\right) \in \mathscr{I}_{m n k}$, provided the intervening matrix bay is intact. In other words, the pair of fibre segments flanking every intact matrix bay is connected by edges, and the pair of fibre segments flanking broken matrix bays are not connected by edges. The vertices and edges of the graph in the $n=0$ longitudinal plane of a composite for the case of an illustrative set of fibre breaks, and matrix failures, are shown in Fig. 7.

At the beginning of a simulation, when all fibre and matrix elements of the simulation cell are intact, $E$ is initialised such that every vertex is connected to its two neighbours along the fibre, and to the six fibre segments neighbouring it in its own block. $E$ is then updated after each fibre break or matrix failure by deleting edges. The presence of a circumnavigating path traversing along edges, starting from a vertex, and ending again at the same vertex indicates that the periodic specimen is not fractured. The absence of any such circumnavigating path indicates that the specimen is fractured.

Before implementing the aforesaid condition to detect specimen fracture, the periodic cell is made artificially non-periodic by replicating one block, say $k=0$ to make a new block, $k=2 K$. During replication, every vertex in block $k$ is copied over to the replica block, $k=2 K$. All the edges connecting any of the vertices in block $k=0$ to block $k=2 K-1$ are deleted. Likewise, all the edges connecting the vertices in block $k=2 K$ to block $k=1$ are also deleted. All the other edges in blocks $k=0$, and $k=2 K$ are preserved. The distance on the graph between vertices $v_{m n 0}$ and $v_{m n, 2 K}$ is then determined using the breadth-first-search algorithm (Cormen et al 2009), for some ( $m, n$ ). A finite distance on the graph indicates the presence of a circumnavigating path, and the non-occurrence of specimen fracture. If an infinite distance occurs, it indicates the non-availability of a circumnavigating path. In this case, the criterion is retested starting from another $v_{m^{\prime} n^{\prime} 0}$. If all the fibres are exhausted, and a circumnavigating path is still not found, the model specimen is algorithmically recognised to be fractured.

## 5 Results

The solution methodology for the shear-lag model presented in Sec. 3 promises an accurate approximation with much smaller computational effort than that based on eigenvector expansions, proposed by Sheikh and Mahesh (2018). The gains in computational speed are quantified for test cases in Sec. 5.1. The correlation between accuracy of the solution and the order of the Chebyshev polynomial used to interpolate the shear stress variation in matrix bays is determined in Sec. 5.2. Overload profiles predicted near single breaks, and small clusters of breaks are then presented in Sec. 5.3. The effect of the periodicity imposed along the fibre direction on stress overloads is then considered in Sec. 5.4. Finally, the computational efficiency of the present method is exploited to study fracture development in large simulation cells in Sec. 5.5.

### 5.1 Computation time

Consider a model composite specimen comprised of $v^{2}=256$ fibres, axially extending $2 K=10$ blocks, each block $\Delta=0.5$ long. A circular cluster of breaks of radius $r \in\{0,1,2, \ldots\}$ is introduced by breaking all the fibres distant no more than $r$ inter-fibre spacings from the fibre $(m, n)=(0,0)$ in the $k=0$ block. Here, and in the sequel, the centre-to-centre distance between two neighbouring fibres in the hexagonal lattice is taken to be unity. Corresponding to $r=0,1$, and 2 , these clusters have $1,1+6=7$, and $1+6+12=19$, fibre breaks, respectively.

The overloads due to the fibre breaks, with and without matrix failure are studied. The condition wherein the matrix bays surrounding the fibre breaks are intact is termed 'no matrix failures'. The condition wherein all the matrix bays surrounding a fibre break are failed is termed 'with matrix failures'. In the latter case, 12 matrix bays, each of length $\Delta$ are failed, six each in blocks $b=0$, and $b=9$. Other matrix bays are assumed intact.

The redistributed tractions in the model specimen can be computed in two ways: First, the eigenvector expansion method of Sheikh and Mahesh (2018), and second, the fast Fourier transform (FFT) based method, developed presently. CPU times corresponding these two calculations performed in the same computer, with and without matrix failure, were measured. The results are shown in Fig. 8. In the case of the eigenvector expansion method, the total computational time includes the times to compute the eigenvalues and eigenvectors of the block matrices, and the time to superpose the various blocks. For parity, the total computational time of the FFT solution also includes the time to compute the unit solutions, the time to determine their weights, and that to superpose the unit fields. No simulation is allowed to run longer than 5000s. In the case of the eigenvector solution method, only the cases $r \in\{0,1,2\}$ complete in fewer seconds. Using the present FFT based method, computations up to $r=7$ complete in under 3 s .

It is clear from Fig. 8 that the present method is orders of magnitude faster than the eigenvector solution method of Sheikh and Mahesh (2018). The computational requirement for the eigenvector solution increases with increasing numbers of fibre breaks, but that of the present method remains practically constant, for fixed $v^{2}$. The latter is to be expected because all the matrix-vector multiplication operations associated with Eqs. (102) and (103) are performed over the entire simulation cell,


Fig. 8: Comparison of the computational (CPU) time required to solve for the elastostatic fields in a simulation cell comprised of $2 K=10$ blocks each of length $\Delta=0.5$, using the (a) eigenvector method of Sheikh and Mahesh (2018), and (b) the present FFT based method.


Fig. 9: Computational time in seconds, to solve for the tractions in a simulation cell comprised of $2 K=10$ blocks each of length $\Delta=0.5$, and $v^{2} \in\left\{2^{4}, 2^{6}, \ldots, 2^{12}\right\}$ fibres, using the the present FFT based method.
regardless of the number of fibre breaks. On the other hand, introduction of the matrix failures causes the computational effort of the FFT method to increase, while the computational time requirement of the eigenvector solutions remains practically the same, with or without matrix failures. This too can be understood by noting that in the present method, matrix failures increase the number of unknowns in Eq. (103), which therefore, adds to the computational effort. In the eigenvector method, matrix failures are absorbed into the governing equations, Eq. (22). The time to extract the eigensolution of the matrix, with or without matrix failures remains the same. Therefore, the presence of matrix failures does not increase the eigensolution-based computational time.

The increase of computational effort associated with solving for the traction redistribution in patches of increasing size, $v^{2} \in\left\{2^{4}, 2^{6}, \ldots, 2^{12}\right\}$, is shown in Fig. 9. For each size, a circular cluster of breaks, of radius $v / 2$ is centered at the fibre $(m, n)=(0,0)$. All the matrix bays surrounding the fibre breaks are assumed failed. As before, matrix failure extends one bundle $(\Delta=0.5)$ to each side from each fibre break. Fig. 9 also plots the curve

$$
\begin{equation*}
\text { CPU time }=\left(2 \times 10^{-4} \mathrm{~s}\right) v^{2} \log \left(v^{2}\right)+2.5 \mathrm{~s}, \tag{106}
\end{equation*}
$$

an analytical function that reasonably fits the computational times. This shows that computational effort actually scales as $O\left(v^{2} \log \left(v^{2}\right)\right)$, as expected in Secs. 3.2, and 3.3.


Fig. 10: The variation with $P$, of the improvement in approximation, $\varepsilon_{P}$, Eq. (107), of normal tractions in all the fibres. $P-1$ denotes the highest polynomial order in the Chebyshev basis.

### 5.2 Accuracy of polynomial expansion

As noted in Sec. 3.2.2, the functional form of the shear traction in a matrix bay is described in terms of Chebyshev polynomials, up to order $P$. It is expected that the exact shear tractions are more closely approached with increasing $P$. To quantify this approach, let $d \tilde{u}_{m n k} / d \zeta(P)$ denote the traction at fibre location ( $m, n, k$ ) calculated by describing the matrix bay shear tractions as sums of Chebyshev polynomials of orders $p \in\{0,1, \ldots P\}$, as in Eq. (55). The incremental improvement in approximation with $P$ is defined as:

$$
\begin{equation*}
\varepsilon_{P}:=\stackrel{\nu-1}{\max } \max _{n=0}^{\nu-1} \max _{n=0}^{2 K-1} \max _{k=0}^{2}\left|\frac{d \tilde{u}_{m n k}}{d \zeta}(P)-\frac{d \tilde{u}_{m n k}}{d \zeta}(P+1)\right| \tag{107}
\end{equation*}
$$

Consider a composite simulation cell comprised of $v^{2}=2^{8}$ fibres, discretised longitudinally into $2 K=10$ blocks, each $\Delta=0.5$ long. Let fibre $(m, n)=(0,0)$ be broken at $\zeta=0$, and let all the twelve matrix bays adjacent to the fibre break be failed over one block. Fig. 10 shows the monotonic decrease of $\varepsilon_{P}$ with $P$. For a nominal tolerance of $\varepsilon=10^{-6}, \varepsilon_{P}<\varepsilon$ for $P \geq 5$, i.e., the Chebyshev polynomials accurately approximate the actual shear stress distribution in matrix bays for $P$ as small as 5. Further results are therefore calculated assuming $P=5$.


Fig. 11: Variation of the normal tractions, $1+\bar{\sigma}_{m n k}^{\mathrm{f}}+\bar{\sigma}_{m n k}^{\mathrm{m}}$, given by Eqs. (64), and (87), with position along the fibres, $\zeta$, over half the length of the simulation cell. These overloads are due to a single broken fibre, and matrix failures around it. Normal tractions in the broken fibre are less than unity, while those in the intact neighbours are more than unity. The three colours correspond to the cases with matrix failure extending $0, \Delta$, and $2 \Delta$ to each side of the fibre break, for $\Delta=0.5$.

### 5.3 Normal tractions due to fibre breaks

Consider a $v^{2}=2^{8}$ fibre simulation cell discretised along the fibre direction into $2 K=10$ blocks, each $\Delta=0.5$ long. The simulation cell extends over $\zeta \in[-L, L]=$ $[-2.5,2.5]$. Let the fibre $(m, n)=(0,0)$ be broken at $\zeta=-L=-2.5$, i.e., at the lower boundary of block $k=0$. Matrix bays not immediately abutting the fibre break are assumed intact. As for the matrix bays abutting the fibre break, three cases are considered: (i) the matrix bays surrounding the break are intact, (ii) the matrix bays surrounding the break in block $k=0$, and in block $2 K-1=9$ are failed, and (iii) in addition to the failed bays in (ii), the matrix bays surrounding fibre $(m, n)=(0,0)$ in blocks $k=1$ and $k=8$ are also failed. There are thus 0,12 , and 24 failed matrix bays in cases (i), (ii), and (iii), respectively. It is recalled that by virtue of the periodicity assumed, blocks $k=0$, and $k=2 K-1=9$ are adjacent to each other; the fibre break is located at their common boundary.

The reloading profile calculated for these damaged configurations, in the broken fibre, and the overload profile in its six neighbours are shown in Fig. 11. The stress profiles are shown over $\zeta \in[-L, 0]$; the profile for $\zeta=[0, L]$ is obtained by reflecting about $\zeta=0$. The overload profiles in the six intact neighbours of the broken fibre are the same, by symmetry. Thus, they coincide in Fig. 11.

The reloading and overload profiles for case (i) coincides with previous results (Hedgepeth and Van Dyke 1967; Mahesh et al 1999). In particular, the peak stress concentration of 1.105 is obtained in the intact fibres surrounding the broken fibre, in the transverse plane of the broken fibre, $\zeta=-L$. The normal traction in the broken fibre increases exponentially, due to the shear stresses transmitted to it by the surrounding matrix bays. It regains much of the traction dropped at the fibre break by about $\zeta=-L+1$. In cases (ii) and (iii), the broken fibre stays unloaded over the length of the matrix failure, $\Delta$, and $2 \Delta$, respectively. Even so, it regains much of the load it dropped over unit length along the fibre direction.

The overload profiles in the neighbouring intact fibres depend on the length of the matrix failure. The maximum overload in cases (ii), and (iii) is smaller than that of case (i). Also, while the maximum overload in case (i) occurs at $\zeta=-L$, that for cases (ii) and (iii) occurs at $\zeta=\Delta$ and $2 \Delta$, respectively. It is concluded that matrix failure reduces the stress concentrations in the neighbours of the broken fibre, and shifts the maximum stress concentration outside the transverse plane of the initial break.

Next, let all the fibres whose centres are located within a closed disc of radius $r=1$ from fibre $(m, n)=(0,0)$ be broken in the common transverse plane $\zeta=-L$. Seven fibre breaks comprise this cluster. The six broken fibres surrounding $(m, n)=$ $(0,0)$ will have the same overload profile, by symmetry. This may, however, differ from that of fibre $(m, n)=(0,0)$ itself. Thus, there will be two distinct reloading profiles in the broken fibres. Twelve intact fibres, arranged along the edges of a hexagon, neighbour the cluster of breaks. Of these, six intact neighbours form the vertices of the hexagon, and six are centered at the mid-points of its sides. By symmetry, the overload profiles in former group, and in the latter group must be the same. Therefore, there will two distinct overload profiles for the neighbouring intact fibres also.

Let cases (i) and (ii) correspond to the case of no matrix failures, and that wherein all the matrix elements abutting any of the failed fibres in blocks $k=0$, and $k=9$ are failed, respectively. The reloading, and overload profiles in the broken and neighbouring intact fibres are shown for cases (i), and (ii) in Figs. 12a and 12b, respectively.

The greatest stress concentration of about 1.4 in the intact neighbours occurs in case (i) in the plane $\zeta=-L$ of the initial breaks, as seen in Fig. 12a. This peak stress concentration is softened by matrix failure, as shown in Fig. 12b. The peak stress concentration decreases, and shifts out of the transverse plane of the initial breaks.

### 5.4 The effect of periodicity

Periodicity along the fibre direction is enforced in the present model composite through Eqs. (14), and (16), thereby making the model fields dependent upon the length of the periodic cell along fibre direction, $2 L$. This dependence is presently studied in a simulation cell comprised of $v^{2}=2^{8}$ fibres discretised into blocks of length $\Delta=0.5$, for $L \in\{1,2.5,5\}$, which correspond to $2 K \in\{4,10,20\}$, respectively. A fibre break is located at $\zeta=-L$ in the fibre $(m, n)=(0,0)$. The overloads developed in the fibres $(m, 0)$, for $m \in\{1,2, \ldots, v / 2\}$ in the plane $\zeta=-L$ are shown in Fig. 13.


Fig. 12: Variation of the normal tractions, $1+\bar{\sigma}_{m n k}^{\mathrm{f}}+\bar{\sigma}_{m n k}^{\mathrm{m}}$, given by Eqs. (64), and (87), with position along the fibres, $\zeta$, over half the length of the simulation cell. These overloads are due to a cluster of broken fibres of unit radius, and matrix failures around it. Matrix failure extending (a) 0 , and (b) $\Delta$ on either side of $\zeta=-L$ are considered. Since matrix failure is suppressed in (a), $\bar{\sigma}_{m n k}^{\mathrm{m}}=0$.


Fig. 13: Variation of the overloads, $\bar{\sigma}_{m n k}^{\mathrm{f}}+\bar{\sigma}_{m n k}^{\mathrm{m}}$, given by Eqs. (64), and (87), due to a single break located at $(m, n)=(0,0)$ and $\zeta=-L$ on the intact fibres $(m, 0)$ located in its transverse plane. Matrix failure extending (a) 0 , and (b) $\Delta=0.5$ either side of $\zeta=-L$ are considered. Since matrix failure is suppressed in (a), $\bar{\sigma}_{m n k}^{\mathrm{m}}=0$.

The overloads, $\bar{\sigma}_{m n k}^{\mathrm{f}}$, developed in the absence of matrix failure are shown in Fig. 13a. The predicted overloads are comparable for $L=2.5$, and $L=5$, and decrease with distance from the fibre break. However, when $L=1$, the overloads are nearly constant for $3 \leq m \leq 8$, simulating equal load sharing conditions. The maximum overload, realised on the fibre $(m, n)=(1,0)$, however, remains nearly constant for all $L$.

The overloads, $\bar{\sigma}_{m n k}^{\mathrm{f}}+\bar{\sigma}_{m n k}^{\mathrm{m}}$, developed in the presence of matrix failure in all the matrix bays abutting the fibre break are shown in Fig. 13a. The matrix failures extend over a distance $\Delta$ in all the cases. Again, the predicted overloads are comparable for $L=2.5$, and $L=5$, and decrease with distance from the fibre break. Also, when $L=1$, the overloads are nearly constant for $3 \leq m \leq 8$. Furthermore, the maximum overload, realised on the fibre $(m, n)=(1,0)$, is substantially smaller when $L=1$, than when $L=2.5$, or $L=5$.

It is concluded that as the specimen length, $L$, is decreased, the periodic boundary conditions cause the localisation of the stress overloads in the vicinity of fibre breaks to diminish. This diminution is enhanced in the presence of matrix failures.

### 5.5 Fracture development

Fracture simulations are performed on the model specimen described in Sec. 2. Simulations are performed for the composite sizes, $v^{2} \in\left\{2^{8}, 2^{10}\right\}$, and for nondimensional longitudinal half-lengths, $L \in\{2.5,5.0,10.0\}$ divided into $2 K \in\{10,20,40\}$ equal blocks respectively, each of length $\Delta=0.5$. Chebyshev expansion of the shear stresses in the matrix bays following Eq. (55) is performed to order $P-1=3$, which corresponds to $\varepsilon_{P} \approx 10^{-6}$ in Fig. 10. The Weibull modulus of the fibres, which appears in Eq. (104), is taken to be $\rho=10$.

Corresponding to each $v$, and $L$, for each of the two rules governing matrix failure, Monte Carlo fracture simulations are performed on $n_{\text {sim }}=256$ model specimen following the algorithm given in Sec. 4. Let $\sigma_{(i)}, i \in\left\{1,2, \ldots, n_{\text {sim }}\right\}$ denote the average load per fibre at the instant of specimen fracture, in the $i$-th specimen, sorted in ascending order, so that $\sigma_{(1)} \leq \sigma_{(2)} \leq \ldots \leq \sigma_{\left(n_{\text {sim }}\right)}$. The empirical probability of specimen failure at stress level $\sigma_{(i)}, G_{\nu^{2} ; L}\left(\sigma_{(i)}\right)$, is then taken to be

$$
\begin{equation*}
G_{v^{2} ; L}\left(\sigma_{(i)}\right)=\frac{i-1 / 2}{n_{\text {sim }}} . \tag{108}
\end{equation*}
$$

$G_{V^{2} ; L}\left(\sigma_{(i)}\right)$ is the empirical distribution of model composite strength. It has been shown by a number of works in the literature (Harlow and Phoenix 1981; Smith 1982; Beyerlein and Phoenix 1997; Landis et al 2000) that it is more insightful to study the strength distribution of the weakest-link, defined as:

$$
\begin{equation*}
W_{v^{2} ; L}\left(\sigma_{(i)}\right)=1-\left(1-G_{v^{2} ; L}\left(\sigma_{(i)}\right)\right)^{1 /\left(2 L v^{2}\right)} \tag{109}
\end{equation*}
$$

Accordingly, Fig. 14 shows the empirical weakest-link strength distributions, $W_{v^{2} ; L}\left(\sigma_{(i)}\right)$, obtained with and without matrix failure on Weibull probability coordinates. In these coordinates, Weibull distributions plot as a straight line. It is clear that none of the empirical weakest-link distributions $W_{v^{2} ; L}\left(\sigma_{(i)}\right)$ is Weibull distributed.

In Fig. 14, the distributions corresponding to the simulation cell sizes, $v^{2}=$ $2^{8}$, and $v^{2}=2^{10}$, with $L=2.5$ assuming no matrix failure are the strongest.


Fig. 14: Empirical weakest link distributions, $W_{\nu^{2} ; L}$ plotted on Weibull probability paper, for various model specimen sizes, corresponding to $\rho=10$. Distributions obtained with and without matrix failure are shown.

They collapse onto each other, i.e., they exhibit weakest-link scaling (Harlow and Phoenix 1981). Although not shown here, the weakest-link distributions for longer model composites, $L=5.0$, and $L=10.0$ with matrix failure suppressed also collapse onto the common master curve.

The scaling obtained in Fig. 14 with matrix failure does not obey weakest-link scaling over the probability range considered. For a fixed number of fibres, $v^{2}$, the empirical weakest-link strength distributions, $W_{v^{2} ; L}\left(\sigma_{(i)}\right)$ move toward obeying the weakest-link scaling with increasing $L$. For fixed $L$ also, weakest-link scaling is better obeyed with increasing $v^{2}$. In both cases, weakest-link scaling is obeyed in the upper tail, but not in the lower tail of the empirical distributions. These observations indicate that larger $v^{2}$, and $L$ are required if the strength distributions are to obey weakest-link scaling with matrix failure than with no matrix failure. They also suggest that the weakest-link event with matrix failure may be more spatially diffuse than that without matrix failure.

It is also observed in Fig. 14 that for fixed $v^{2}$, the strength of the median specimen decreases with increasing model length, $L$. Further, the slope of $W_{v^{2} ; L}\left(\sigma_{(i)}\right)$ with matrix failure is greater than that without matrix failure, indicating that the former strengths are less scattered than the latter. These observations can be understood on the basis of the results in Sec. 5.4. With increasing $L$, the stress overloads due to a fibre break, or a small cluster of breaks, is more strongly localised over its neighbouring intact fibres, and thereby reduces the mean strength. On the other hand, reducing $L$ makes the load sharing more equal amongst intact


Fig. 15: Normalised stress-strain curves of the median $v^{2}=2^{10}, L=2.5, \rho=10$ specimen. Separate curves are obtained with and without matrix failure.
fibres, which is associated with smaller scatter in the composite strengths (Mahesh et al 1999).

As noted in Sec. 4, during the fracture simulations, the average load per fibre is monotonically increased. The normalised stress-strain plots for the median specimen amongst the $n_{\text {sim }}$ specimens with $v^{2}=2^{10}, L=2.5$ simulated with and without matrix failure are shown in Fig. 15. For this size of the model composite, $\psi \approx 2.3737$ in Eq. (105). The stress-strain graph for specimen with matrix failure visibly deviates from linearity, whereas that of the specimen with matrix failure suppressed remains nearly linear until fracture.

The last load increase usually occurs well before the fracture of the specimen. The step at which the last load increment is applied is termed the critical step, and is marked as such in Fig. 15. Large strains develop between the critical step and the final step at which fracture occurs without any load increase. The curves in Fig. 15 are not shown up to the fracture strain in order to clearly reveal the shape of the stress-strain curve up to the critical step.

Fig. 16 shows the pattern of fibre breaks in all the blocks at the critical step in the median $v^{2}=2^{10}, L=2.5, \rho=10$ specimen, with matrix failure suppressed. A broken fibre at the bottom of block $k$ is indicated by a dot at the fibre's location. Fig. 17 shows the state of fibre breakage after the macroscopic fracture of the same specimen is realised by the breakage of all the fibres in the $k=4$ transverse section.

A fibre break located at ( $m, n, k$ ) shields other fibre locations ( $m, n,[k \pm p]$ ) from breaking. This is because the normal traction builds up in the broken fibre gradually with distance from the break, and thereby unloads other locations in the


Fig. 16: Distribution of fibre breaks (red dots) at the critical step, when the maximum average load is first reached, in the median $v^{2}=2^{10}, L=2.5, \rho=10$ specimen without matrix failure. The state of damage in all the blocks $k \in\{0,1, \ldots, 9\}$ is shown.


Fig. 17: Distribution of fibre breaks (red dots) at the final step corresponding to macroscopic failure in the median $v^{2}=2^{10}, L=2.5, \rho=10$ specimen without matrix failure. The state of damage in all the blocks $k \in\{0,1, \ldots, 9\}$ is shown.
same fibre, as seen in Sec. 5.3. This effect is called shielding. Fibres are unlikely to break at shielded sites. In Fig. 17, a cluster of shielded sites, devoid of fibre breaks is located near the top edge, $n=v$, of the simulation cell in the blocks $k \in\{0,1,2,3,5, \ldots, 9\}$. These sites are shielded by the fibre breaks in the $k=4$ block. Based on the locations of the shielded sites in Fig. 17, it is inferred that specimen fracture developed from the cluster of breaks labelled $C$ in the $k=4$ block in Fig. 16. The $W_{v^{2} ; L}\left(\sigma_{(i)}\right)$ distributions of Fig. 14 thus correspond to the probability of occurrence of this event.

Turning next to the case with matrix failures, Fig. 18 shows the pattern of fibre breaks (red dots), and matrix failure (purple lines) at the critical step in the median $v^{2}=2^{10}, L=2.5, \rho=10$ model composite. At the critical step, a number of fibre breaks and matrix failures in the bays abutting the fibre breaks are observed. Again, it is not evident which of these will propagate catastrophically. The fibre breaks, and matrix failures after the formation of a macroscopic crack, following the criterion of Sec. 4 is shown in Fig. 19. It is seen that no one transverse plane has fully failed; the fracture surface threads across multiple transverse planes connected by matrix failures. This is consistent with the brush-like fracture surface of glass fibre reinforced composites observed experimentally (Hull and Clyne 1996).

The shielded fibre locations in Fig. 19 enable the identification of the fibre break clusters that propagated to produce the final crack. In Fig. 19, one set of shielded sites, devoid of fibre breaks is located near the top edge, $n=v$, of the simulation cell in the blocks $k \in\{1, \ldots, 8\}$. These sites appear to be shielded by the cluster of fibre breaks, and matrix failures, labelled $C_{1}$, extending over the $k=0$, and $k=9$ blocks. It is recalled that these blocks are adjacent to each other due to the periodic boundary conditions.

Cluster $C_{1}$ is not the only nucleus of the catastrophic crack. Additional nuclei are indicated by other shielded sites. One such set of shielded sites is observed near the bottom edge, $n=0$, in blocks $k \in\{0,3, \ldots, 9\}$. These fibre locations are shielded by the cluster of fibre breaks, and matrix failures in the blocks $k=1$, and $k=2$, labelled $C_{2}$ in Fig. 18. These considerations show that the fracture of the median specimen with matrix failure occurs by the growth of multiple fibre break and matrix failure clusters and their coalescence. These observations suggest that the reason that $W_{\nu^{2} ; L}\left(\sigma_{(i)}\right)$ in Fig. 14 does not obey weakest-link scaling is that fracture does not develop from a single nucleus.

It was observed previously that with increasing $v^{2}$, and $L, W_{v^{2} ; L}\left(\sigma_{(i)}\right)$ increasingly obey weakest link scaling. This suggests that with increasing $v^{2}$, and $L$, the dominant failure mode must switch from a global failure mode involving the coalescence of clusters of fibre breaks and matrix failure to the catastrophic propagation of a single cluster.

## 6 Discussion

In the classical shear lag models (Cox 1952; Hedgepeth 1961), fibres are assumed to be loaded in simple tension, and matrix elements in simple shear. These models assume a perfectly undamaged matrix, and treat fibre breaks as the only type of progressing damage. In the linear shear-lag model developed here, both fibre and matrix failures are admitted as microscopic damage modes. Fibre and matrix


Fig. 18: Distribution of fibre breaks (red dots), and matrix failures (purple lines) at the critical step, when the maximum average load is first reached, in the median $v^{2}=2^{10}, L=2.5, \rho=10$ specimen with matrix failure. The state of damage in all the blocks is shown.


Fig. 19: Distribution of fibre breaks (red dots), and matrix failures (purple lines) at the final step corresponding to macroscopic fracture, in the median $v^{2}=2^{10}$, $L=2.5, \rho=10$ specimen with matrix failure. The state of damage in all the blocks is shown.
failure are regarded as sites of zero tensile traction, and zero shear traction, respectively. The model so developed can be solved inefficiently in $O\left(N^{3}\right)$ time, by a straightforward extension of the classical algorithms, where $N$ is the number of fibre segments in the model. A faster $O(N \log N)$ algorithm exploiting the translation invariance of the elastostatic fields of unit failure events is presently proposed. For given computer hardware, the present algorithm can solve for the elastostatic state in much larger simulation cells.

The present model admits straightforward extensions beyond the development given here. For example, the present development assumes a rhombus-shaped cross-section comprised of $v$ fibres along both the $m$ - and $n$-axes in Sec. 2.1. This can be readily generalised to non-rhombus parallelogram shaped cross-sections comprised of different number of fibres, along the $m$ - and $n$-axes, as in Curtin (2000). Also, the $A_{m n p q}$ matrix in Eq. (5) could be modified to signify interactions further afield than the nearest ring of neighbours. Further, the entire analysis could be carried out, with considerable simplification, for a two-dimensional tape in the $m-z$ plane (Gupta et al 2017). Finally, accounting for the tensile loads carried by the matrix, and accounting for matrix cracks also seems feasible, following the approach suggested by Beyerlein and Landis (1999).

Other seemingly natural extensions of the present model are not straightforward. For example, implementing a pull-out stress, $\tau \neq 0$ over a varying length that restores the far-field load in a broken fibre, as in Ibnabdeljalil and Curtin (1997), is not straightforward. This is because the average load experienced by the composite determines the length of the sliding zone. Since the configuration of the damaged regions depends upon the applied load, the problem ceases to be linear, and the present methods do not apply. Also, the model cannot be readily extended to account for infinite length along the fibre direction. This is because translation invariance along the fibre direction, and therefore the applicability of FFT-based algorithm, break down in the absence of periodicity.

The present work focuses on linear shear-lag models, wherein the influence of fibre breaks can be superposed. Linear models represent highly idealised limiting cases of detailed non-linear models, which account for a variety of non-linear mechanisms including matrix plasticity, interfacial debonding, and frictional sliding across the debonded interface (Landis and McMeeking 1999; Okabe et al 2001; Xia et al 2002; Mishnaevsky Jr and Dai 2014; Swolfs et al 2013; Mishra and Mahesh 2017). The non-linearity disallows the superposition of break influences in these models. Therefore, they are much more computationally expensive to solve. Computationally tractable simulation cells of these models tend to be small. For example, the largest patches simulated by Mahesh and Mishra (2018) were comprised only of 128 fibres, and extended only one characteristic length along the fibre. The fast Fourier transform based solution methods developed here will not carry over to these complex models.

## 7 Conclusions

A linear, periodic, three-dimensional shear-lag model for a polymer matrix composite has been developed, which allows damage in the form of fibre breaks, and matrix failure. A fast Fourier transform based method has been proposed to obtain the elastostatic state in this composite. The present algorithm is orders of mag-
nitude faster than than the classical approach. Fracture simulations have been performed using the fast algorithm. It is shown that even though the matrix carries no tensile loads, matrix failure affects the stress redistribution in partially failed model composites, and alters the development of fracture. This establishes the importance of accounting for matrix failure while assessing the reliability of composite materials.

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